MAS275 Probability Modelling Example 21: Monopoly

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Each player has a token which moves round the board in a circuit. On each turn the initial movement is determined by moving two dice.
Monopoly simplified

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One corner (square 10) is labelled “Jail” and the opposite corner (square 30) “Go to Jail”.

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Let $X_n$ be the $n$th square visited by the player. The player starts on square 0 (or 40; labelled “Go”), so $X_0 = 0$. 

In most cases, if on square $j$, the player moves to square $j + k \mod 40$ with probability given by $d_k$, where $d_k$ is the probability of $k$ being the sum of the numbers on the two dice. (E.g. $p_{5,11} = d_6 = \frac{5}{36}$, $p_{37,8} = d_{11} = \frac{2}{36}$.)

If the player lands on square 30 (“Go to Jail”) the token moves to square 10 (“Jail”). So we set $p_{30,10} = 1$ (and $p_{30,i} = 0$ for $i \neq 10$).
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Jail high probability; squares between about 15 and 31 (including orange, red, yellow sets) have higher probabilities than others.
Adding Chance and Community Chest

(Based on US edition circa 1983)
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Then \( p_{7,i} \) is \( \frac{6}{16} d_{i-7} \) plus the probability of being sent to \( i \) by the Chance card, which is \( 1/16 \) for squares 0 (Go), 4 (back 3 spaces), 5, 10 (Jail), 11, 12 (nearest utility, Electric Company), 24 and 39, and \( 1/8 \) for square 15 (nearest railway, because there are two of these cards).
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Similar changes to rows 22 and 36
Landing on squares 2, 17 and 33 gives “Community Chest”. Similar analysis to Chance, but only two cause a further move, one to Jail and one to Go.
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Then $p_{2,i}$ is $\frac{14}{16} d_{i-2}$ plus the probability of being sent to $i$ by the Chance card, which is 1/16 for squares 0 (Go) and 10 (Jail). Similar changes to rows 17 and 33.
With the new Markov model, we get the following stationary distribution:
New stationary distribution

Properties on the side of the board after Go (squares 1-9) have relatively low probabilities, except square 5, which is a railway and benefits from Chance cards sending tokens there.
Square 10 (jail) still has the highest probability. Square 0 or 40 (Go) is second highest.
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Property with highest probability is square 24, Illinois Avenue (US edition), one of the red group. The other properties in this area, the orange and red groups, also have relatively high probabilities.
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Chance and Community Chest cards are used in order, not shuffled. (This is non-Markov.)