Spectra of adjacency matrices of random geometric graphs

Paul Blackwell, Mark Edmondson-Jones and Jonathan Jordan

University of Sheffield *

22nd December 2006

Abstract

We investigate the spectral properties of the adjacency matrices of random geometric graphs both theoretically and by simulation, concentrating on the thermodynamic limit. Our results show interesting differences from those previously found for other models of random graphs. In particular the spectra do not show the symmetry about 0 found for classical and scale-free random graphs, and we find a striking singularity at -1.[1]

1 Introduction

In [3] the eigenvalues of adjacency matrices of several different models of random graphs are discussed, largely through calculation of the eigenvalues of simulated graphs. The models of random graphs discussed are classical (Erdős-Rényi) random graphs, Watts-Strogatz small world graphs (as introduced in [13]) and scale-free graphs (as described in [1]). The simulation results show interesting differences between the spectra of the adjacency matrices of the different classes of graph. In [2] it is proposed to use the spectrum of the adjacency matrix as a tool for indicating which model may be most appropriate for a particular real-world graph.

One model of graph not discussed in [3] is the *random geometric graph* extensively discussed in the book [10]. Random geometric graphs are constructed by taking a random configuration of points in space, considering the points as vertices of the graph and adding edges between those points which are less than some threshold distance r apart, and hence have a spatial element absent from Erdős-Rényi and scale-free graphs. Examples of motivating applications from the introduction to [10] include communication networks, biological networks and applications to multivariate statistics.

*Department of Probability and Statistics, University of Sheffield, Hicks Building, Sheffield S3 7RH, UK

[1] AMS 2000 Subject Classification: Primary 05C80, secondary 60D05.

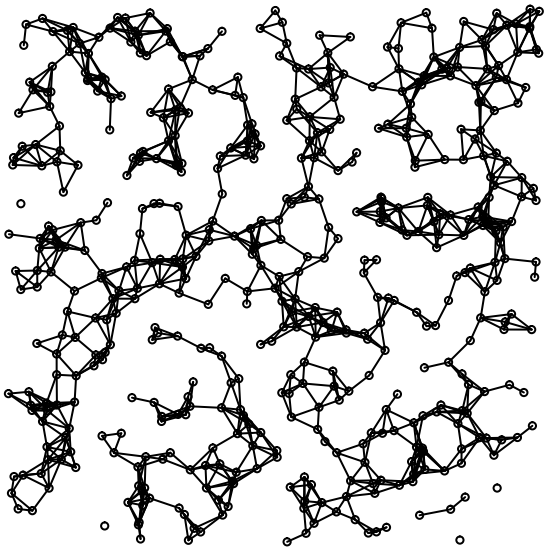

Figure 1: An example of a random geometric graph

We investigate the spectra of adjacency matrices of these graphs in two ways. In section 3, we show some results about the behaviour of the spectrum in the so-called thermodynamic limit, using techniques from [10], and we compare these results to those for Erdős-Rényi random graphs. In section 4, we obtain some simulated spectra of adjacency matrices of large random geometric graphs, and compare them to the simulated spectra for the other models found in [3].

2 Definitions and notation

2.1 Random geometric graphs

We base our notation on [10].

Given a finite set $\mathcal{X} \subset \mathbb{R}^d$, a norm $\|\cdot\|$ on $\mathbb{R}^d$ and $r \in \mathbb{R}^+$, we define $G(\mathcal{X}; r)$ to be the (non-directed) graph whose vertex set is given by $\mathcal{X}$ and which has an edge between X and Y if and only if $\|Y - X\| \leq r$.

We take a probability density function f on $\mathbb{R}^d$, and let $\{X_i; i \in \mathbb{N}\}$ be a family of independent and identically distributed random variables on $\mathbb{R}^d$ with common density f. We then let $\mathcal{X}_n = \{X_1, \ldots, X_n\}$, and we define the *random geometric graph* $G(\mathcal{X}_n; r)$. As in [10], we will assume that f is bounded, and let $f_{\max}$ be its essential supremum.

Interest is largely in asymptotic properties of sequences $G(\mathcal{X}_n; r_n)$ for some sequence $\{r_n; n \in \mathbb{N}\}$. In [10], two particular limiting regimes are introduced: the *thermodynamic limit* with $nr_n^d \to \rho \in (0, \infty)$, so that the expected degree of a vertex tends to a constant, and the *connectivity regime* with $\left(\frac{n}{\log n}\right) r_n^d \to \alpha \in (0, \infty)$. In this paper, we will concentrate on the thermodynamic limit.

Let $B(x;r)$ be the ball of radius r around x. Then define $V(y_1,\ldots,y_m)$, as in [10], p50, to be the Lebesgue measure of $\bigcup_{i=1}^m B(y_i,1)$.

We will need the concept of a *feasible* graph, for a given probability density function f on $\mathbb{R}^d$. As defined in [10], p47, a graph Γ with k vertices is feasible if $\mathbb{P}(G(\mathcal{X}_k;r)\cong\Gamma)>0$ for some r; that is that for some r there is a positive probability that the random geometric graph on k vertices is isomorphic to Γ.

2.2 Spectra of graphs

Given a (non-directed) graph G with vertex set V and no multiple edges, the adjacency matrix of G is a square matrix A, with rows and columns indexed by V, such that A_{uv} is 1 if there is an edge between u and v and 0 otherwise. As in [3, 6], we will be interested in the eigenvalue spectrum of the adjacency matrix.

Given a graph G, define the spectral density ρ_G of its adjacency matrix to be a measure on $\mathbb{R}$ with total weight 1 and equal weight on each eigenvalue of its adjacency matrix. We will be particularly interested in considering sequences of graphs $\{G(\mathcal{X}_n;r_n);n\in\mathbb{N}\}$ and looking at limiting properties of $\rho_{G(\mathcal{X}_n;r_n)}$ as $n\to\infty$. One question is whether there exist singularities in the limiting behaviour, such as values λ such that $\rho_{G_n}(\{\lambda\})\not\to 0$ as $n\to\infty$. The existence of such singularities is often associated with the existence of localised eigenvectors with eigenvalue λ, that is eigenvectors which take zero values on all but a small subset of the vertices. Examples of random graph models where this occurs include sparse Erdős-Rényi graphs, [3], and random trees, [9].

Another matrix associated with a graph is the Laplacian, and eigenvalues of the Laplacian are the subject of [5]. In this paper we will concentrate on the adjacency matrix; however, some similar results will apply to the Laplacian.

3 The spectral density in the thermodynamic limit

In this section we look at the behaviour of the spectral density $\rho_{G(\mathcal{X}_n;r_n)}$ as $n\to\infty$ in the thermodynamic limit. We will show (Corollary 6) that the spectral densities converge to a limit, and we will also show that this limiting density has singularities due to localised eigenfunctions associated with structures present in the graph.

We will start by investigating certain types of subgraph responsible for localised eigenfunctions of the adjacency matrix, and their prevalence in the thermodynamic limit, where $nr_n^d\to\rho\in(0,\infty)$.

First of all we show that results in [10] imply fairly straightforwardly that there will be singularities in the limiting behaviour of the spectrum of the adjacency matrix of $G_{\mathcal{X}_n;r_n}$ in the thermodynamic limit. As in [10], p50, we define $p_\Gamma(\lambda)$ by

$$\frac{\lambda^{k-1}}{(k-1)!}\int_{(\mathbb{R}^d)^{k-1}} h_\Gamma(\{\mathbf{0},x_1,\ldots,x_{k-1}\}) \exp(-\lambda V(\mathbf{0},x_1,\ldots,x_{k-1})d(x_1,\ldots,x_{k-1}).$$

Proposition 1. *Given a feasible graph Γ, with $p_\Gamma(\lambda)>0$ for all $\lambda>0$, if μ is an eigenvalue of the adjacency matrix of Γ then the expected value of $\rho_{G(\mathcal{X}_n;r_n)}(\{\mu\})$ will*

be at least $k^{-1}\int_{R^d} p_\Gamma(\rho f(x))f(x)dx$. *Furthermore*

$$\liminf_{n\to\infty} \rho_{G(\mathcal{X}_n;r_n)}(\{\mu\}) \geq k^{-1}\int_{R^d} p_\Gamma(\rho f(x))f(x)dx,$$

almost surely.

Proof. Proposition 3.3 of [10] shows that the number of separate components of $G(\mathcal{X}_n; r_n)$ isomorphic to Γ, $J_n(\Gamma)$, satisfies

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}(J_n)}{n}\right) = k^{-1}\int_{R^d} p_\Gamma(\rho f(x))f(x)dx,$$

and Theorem 3.15 of [10] shows that J_n/n converges almost surely to $k^{-1}\int_{R^d} p_\Gamma(\rho f(x))f(x)dx$. These components will each give an eigenvalue μ, hence the result. □

We aim to show that there are also singularities which are not due to separate components.

Let Γ be a feasible graph with $k = k_1 + k_2$ vertices, partitioned into two subsets V_1 and V_2, with $|V_i| = k_i$, and such that any graph homomorphism of G fixes V_1 and V_2. We are interested in induced subgraphs of the random geometric graph $G(\mathcal{X}_n, r_n)$ which are isomorphic to Γ with the additional property that vertices in the subset mapping to V_1 are not connected to any vertices outside the subgraph. Using a notation based on that in Section 3 of [10], we set H_n to be the number of such subgraphs in $G(\mathcal{X}_n; r_n)$, and, given a subset A of $\mathbb{R}^d$, $H_{n,A}$ to be the number of such subgraphs for which the left-most point of the vertex set (as defined in [10], p48) is in A.

We base the following definitions on those of $I_n(x_1,\ldots,x_k)$, $h_\Gamma(\mathcal{Y})$ and $p_\Gamma(\lambda)$ in [10]. Let

$$\tilde{I}_n(x_1,x_2,\ldots,x_k) = \int_{\bigcup_{j=1}^{k_1} B(x_j;r_n)} f(x)dx.$$

Let $\tilde{h}_\Gamma(\mathcal{Y}_1,\mathcal{Y}_2)$ be the indicator function of the event that $G((\mathcal{Y}_1\cup\mathcal{Y}_2);1)$ is isomorphic to Γ with $\mathcal{Y}_i$ mapping to V_i. Similarly let $\tilde{h}_{\Gamma,n,A}(\mathcal{Y}_1,\mathcal{Y}_2)$ be the indicator function of the event that $G((\mathcal{Y}_1\cup\mathcal{Y}_2);r_n)$ is isomorphic to Γ with $\mathcal{Y}_i$ mapping to V_i, and that the left-most point of $\mathcal{Y}_1\cup\mathcal{Y}_2$ is in A.

Finally, we set $\tilde{p}_\Gamma(\lambda)$ to be equal to

$$\frac{\lambda^{k-1}}{(k-1)!}\int_{(\mathbb{R}^d)^{k-1}} \tilde{h}_\Gamma(\{\mathbf{0},x_1,\ldots,x_{k_1-1}\},\{x_{k_1},\ldots,x_{k-1}\}) \exp(-\lambda V(\mathbf{0},x_1,\ldots,x_{k-1})d(x_1,\ldots,x_{k-1}),$$

noting that this will be positive for all $\lambda > 0$ if $\tilde{h}_\Gamma(\{\mathbf{0},x_1,\ldots,x_{k_1-1}\},\{x_{k_1},\ldots,x_{k-1}\}) = 1$ on a set of positive Lebesgue measure in $\mathbb{R}^{k-1}$.

Proposition 2. *Suppose that $A\subseteq\mathbb{R}^d$ is open with Leb$(\partial A)=0$, that Γ is a feasible graph with $k=k_1+k_2$ vertices, partitioned into two subsets V_1 and V_2, with $|V_i|=k_i$, and such that any graph homomorphism of G fixes V_1 and V_2, and that $nr_n^d\to\rho\in(0,\infty)$. Then*

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}(H_{n,A})}{n}\right) = k^{-1}\int_A \tilde{p}_\Gamma(\rho f(x))f(x)dx.$$

Further, $\left(\frac{\mathbb{E}(H_{n,A})}{n}\right)$ converges to $k^{-1}\int_A \tilde{p}_\Gamma(\rho f(x))f(x)dx$, almost surely.

Proof. This is closely based on the proofs of Proposition 3.3 and Theorem 3.15 of [10].

We have

$$\begin{aligned} n^{-1}\mathbb{E}(H_{n,A}) &= n^{-1}\binom{n}{k}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}\tilde{h}_{\Gamma,n,A}(\{x_1,\dots,x_{k_1}\},\{x_{k_1+1},\dots,x_k\}) \\ &\quad\times(1-\tilde{I}_n(x_1,x_2,\dots,x_k))^{n-k}f(x_1)^k\prod_{i=1}^k dx_i \\ &\quad+n^{-1}\binom{n}{k}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}\tilde{h}_{\Gamma,n,A}(\{x_1,\dots,x_{k_1}\},\{x_{k_1+1},\dots,x_k\}) \\ &\quad\times(1-\tilde{I}_n(x_1,x_2,\dots,x_k))^{n-k}\left(\prod_{i=1}^k f(x_i)-f(x_1)^k\right)\prod_{i=1}^k dx_i. \end{aligned}$$

Changing variables $y_i = r_n^{-1}(x_i - x_1)$ for $2 \le i \le k$, the first term is asymptotic to

$$\frac{\rho^{k-1}}{k!}\int_{\mathbb{R}^d}\int_{(\mathbb{R}^d)^{k-1}}\tilde{h}_{\Gamma,n,A}(\{x_1,x_1+r_ny_2\dots,x_1+r_ny_{k_1}\},\{x_1+r_ny_{k_1+1},\dots,x_1+r_ny_k\})$$
$$f(x_1)^k\exp\{(n-k)\log(1-\tilde{I}_n(x_1,x_1+r_2y_2\dots,x_1+r_ny_k))\}d(y_2,\dots,y_k)dx_1.$$

Now, as in [10],

$$r_n^{-d}\tilde{I}_n(x_1,x_1+r_2y_2\dots,x_1+r_ny_k)\to f(x_1)V(\mathbf{0},y_2,\dots,y_k),$$

and $\tilde{h}_{\Gamma,n,A}(\{x_1,x_1+r_ny_2\dots,x_1+r_ny_{k_1}\},\{x_1+r_ny_{k_1+1},\dots,x_1+r_ny_k\})$ converges to

$$h_\Gamma(\{\mathbf{0},y_2,\dots,y_{k_1}\}),\{y_{k_1+1},\dots,y_k\}$$

if $x_1 \in A$ and to 0 if $x_1 \notin A \cup \partial A$. Furthermore, the right hand side of our expression for $n^{-1}\mathbb{E}(H_{n,A})$ converges to 0 by the same argument as in the proof of Proposition 3.3 of [10].

Hence the limit is

$$\frac{\rho^{k-1}}{k!}\int_{\mathbb{R}^d}\int_{(\mathbb{R}^d)^{k-1}}h_\Gamma(\{\mathbf{0},y_2,\dots,y_{k_1}\},\{y_{k_1+1},\dots,y_k\}$$
$$f(x_1)^k\exp\{-\rho f(x_1)V(\mathbf{0},y_2,\dots,y_k)I_A(x_1)d(y_2,\dots,y_k)dx_1,$$

giving the first result.

The proof of the almost sure convergence is the same as in the proof of Theorem 3.15 in [10] for components.

□

Proposition 3. *Let G be any graph. If G includes n_1 mutually adjacent vertices sharing the same closed neighbourhood, then the spectrum of the adjacency matrix of G contains the eigenvalue -1 with multiplicity $n_1 - 1$.*

Proof. Without loss of generality we label the mutually adjacent vertices $V_1, V_2, \dots, V_{n_1}$. Clearly the subgraph induced by $V_1, V_2, \dots, V_{n_1}$ is a complete graph on n_1 vertices, K_{n_1}. The other vertices in the shared closed neighbourhood are labelled $V_{n_1+1}, \dots, V_{n_1+n_2}$

, again without loss of generality. The characteristic polynomial can therefore be calculated from

$$
\begin{aligned}
|A-\lambda I_n| &= \begin{vmatrix}
-\lambda & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots \\
1 & -\lambda & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \\
1 & 1 & \cdots & -\lambda & 1 & \cdots & 1 & 0 & \cdots \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots &
\end{vmatrix} \\
&= \begin{vmatrix}
-1-\lambda & 0 & \cdots & 1+\lambda & 0 & \cdots & 0 & 0 & \cdots \\
0 & -1-\lambda & \cdots & 1+\lambda & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \\
1 & 1 & \cdots & -\lambda & 1 & \cdots & 1 & 0 & \cdots \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots &
\end{vmatrix}
\end{aligned}
$$

(by subtracting row n_1 from each of the first $n_1 - 1$ rows)

$$
= (1+\lambda)^{n_1-1} \begin{vmatrix}
-1 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots \\
0 & -1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \\
1 & 1 & \cdots & -\lambda & 1 & \cdots & 1 & 0 & \cdots \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots &
\end{vmatrix}
$$

From this calculation we can see that this adjacency configuration results in a -1 eigenvalue with multiplicity $n_1 - 1$. Clearly the multiplicity of this eigenvalue can be further incremented by other features of the graph (for example from other such groups of vertices). □

We can now show that, in the thermodynamic limit, there is a singularity at -1 which is not due to the effect of isolated components.

Corollary 4. *Given a sequence of random geometric graphs $G(\mathcal{X}_n; r_n)$ satisfying the thermodynamic limit, the limiting behaviour of the spectral density $\rho_{G(\mathcal{X}_n;r_n)}$ involves a singularity at -1 due to structures of the form described in Proposition 3.*

Proof. We consider a graph Γ whose vertex set is partitioned into two sets V_1 and V_2 such that the induced subgraph on V_1 is complete. This will work as long as $\tilde{p}_\Gamma(\lambda) > 0$ for all $\lambda > 0$, which it will be for example if $|V_2| = 2$ and the vertices in V_2 are not connected. We then apply Proposition 2.

□

We now look at the moments of the spectral density for random geometric graphs. This will enable us to show that the spectral densities converge as $n \to \infty$, and also to show certain properties of the limiting density, in particular that it has a positive skewness.

Lemma 5. *For $k \geq 3$, let $M_{k,n}$ be the kth moment of the spectral density $\rho_{G(\mathcal{X}_n;r_n)}$ of the adjacency matrix of $G(\mathcal{X}_n; r_n)$, i.e. $M_{k,n} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^k$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix. Then, as $n \to \infty$ in the thermodynamic limit, $M_{k,n} \to \alpha_k$ almost surely, for some positive constant α_k.*

Proof. This relies on the fact that $M_{k,n} = \frac{1}{n} D_{k,n}$, where $D_{k,n}$ is the number of directed paths of length k in the graph that return to their starting point (see [3], section III.A.).

If Γ is a graph consisting of a path of length k returning to its starting point (perhaps involving traversing an edge of Γ more than once, so Γ may have fewer than k edges) then we get a contribution of $\kappa_\Gamma \tilde{G}_n(\Gamma)$ (where $\tilde{G}_n(\Gamma)$ is the total number of subgraphs of $G_{\mathcal{X}_n;r_n}$, as defined in [10], p47) to $D_{k,n}$, and κ_Γ is a constant depending on Γ. (If Γ is a simple k-cycle with no common vertices, $\kappa = 2k$.) To get $D_{k,n}$ we need to sum $\kappa_\Gamma \tilde{G}_n(\Gamma)$ over the possible Γ.

Hence, by Theorem 3.17 in [10] and the relationship between the asymptotics for $\tilde{G}_n$ and G_n ([10], p47), $M_{k,n} \to \alpha_k$ almost surely, where α_k is bounded below by μ_Γ (where μ_Γ is defined in [10], p48) for any Γ containing a path of length k returning to its starting point.

To show that $\alpha_k > 0$, we only need to show the existence of a graph Γ, containing a path of length k returning to its starting point, which is feasible with $\mu_\Gamma > 0$. As a complete graph K_k satisfies this condition, we have the result.

□

Corollary 6. *The spectral densities $\rho_{G(\mathcal{X}_n;r_n)}$ converge, almost surely, to a limiting spectral density ρ_∞ with moments α_k.*

Proof. To show this we need to show that the sequence of moments α_k is such that there is only one distribution with these moments. A sufficient criterion is that $\limsup_{k\to\infty} \alpha_{2k}^{1/2k}/2k = r < \infty$ ([8], p110).

To show this, we note that any graph Γ consisting of a path of length k returning to its starting point contains a path with l edges, for some $l \leq k-1$, and hence the contribution to $D_{k,n}$ can be bounded above by a multiple of the number of paths with l edges.

The probability that an ordered set of $l+1$ vertices (of which there are $\binom{n}{l+1}$) form a path with l edges can be bounded above by $(f_{\max} \operatorname{vol}(B(0;r_n))^l$. In the thermodynamic limit this will be bounded above by $(C/n)^l$, for some constant C, as $n \to \infty$, and hence the expected number of such paths in $G(\mathcal{X}_n; r_n)$ is bounded above by C^l. This is enough to show that the criterion for the moment sequence is met.

□

Corollary 7. *The spectral densities of $G(\mathcal{X}_n; r_n)$ have a skewness which converges, almost surely, to a positive constant.*

Proof. This follows immediately from the $k = 3$ case of Lemma 5. □

We now compare our results to those for classical (Erdős-Rényi) random graphs $\mathcal{G}(n, p_n)$, the subject of [4], choosing a natural limiting regime to compare to the thermodynamic limit for random geometric graphs. This is where the expected vertex degree converges to a constant, which gives $p_n \sim c/n$ for some $c \in (0, \infty)$. This is briefly discussed in section IV.A of [3], and relevant results about components of the graphs are found in Chapters 4 and 5 of [4].

In contrast to Corollary 7, in the Erdős-Rényi case $\mathcal{G}(n, p_n)$ with $np_n \to c$ as $n \to \infty$, the normalised (by dividing by n) number of cycles of length k in the graph converges to 0. Hence, as stated in [3], the normalised number of loops of odd length converges to 0, and so the odd moments of the spectral density converge to 0, giving spectral densities which are close to symmetric about 0 for large n.

Similarly the cycles present in the graphs of the form that leads to the -1 eigenvalues in Proposition 3 will cause the normalised number of these subgraphs to converge to 0, and hence the result of Corollary 4 does not apply.

4 Simulation

In this section we will concentrate on random geometric graphs on $\mathbb{R}^2$ where the underlying probability density function is uniform on the unit square, i.e.

$$f(x) = \begin{cases} 1 & \text{if } x \in [-1/2, 1/2]^2 \\ 0 & \text{otherwise.} \end{cases}$$

The simulation of the graphs and the calculation of eigenvalues were performed using R [12]. The spectral densities of simulated graphs were plotted using the kernel density estimation function in R, with a triangular kernel and the bandwidth adjusted to avoid over-smoothing.

Figure 2 shows mean spectra for four cases which give the same average degree, 2.5, with increasing n, the number of vertices. (Note that keeping average degree constant while increasing n will give the thermodynamic limit as $n \to \infty$.) There are notable spikes at several values, in particular -1, 0 and 1. Some of these are likely to be due to the effect of small components, as predicted by Proposition 1. In particular, the adjacency matrix of a graph with isolated vertices will have eigenvectors with eigenvalue 0 localised on those isolated vertices. Evidence for singularities due to isolated components is also found for sparse Erdős-Rényi random graphs in [3].

Figure 3 shows a similar plot of four cases with average degree 37.6. We see an obvious spike at -1, as predicted by Corollary 4. The mean proportion of -1 eigenvalues was 0.0659 for 100 vertices, 0.0296 for 500 vertices, 0.0231 for 2000 vertices and 0.0206 for 5000 vertices. Apart from this spike, the central parts of the mean spectra show a roughly triangular shape with a peak at -1, in contrast to the near semi-circular form found for Erdős-Rényi graphs, and a long tail towards the maximal eigenvalue. As predicted by Corollary 7 the mean spectra do not show symmetry about 0.

Apart from the spike at -1 there are no other obvious singularities, presumably because with this relatively high average degree there are very few isolated components. In particular we found no 0, 1 or 2 eigenvalues.

Figure 4 shows mean spectra for four different values of r for fixed $n = 1000$. In each case we see a peak in the spectrum and an apparent singularity at -1. The larger values of r appear to give spectra which are close to symmetric about -1 while the smaller values show a longer positive tail.

We also simulated random geometric graphs on the unit sphere, which avoids edge effects found with graphs on the unit square. The resulting mean spectral densities (not shown here) were very similar, both in terms of the behaviour at -1 and the overall shape of the spectrum.

We note a number of differences in these plots compared with those found in [3] for other graph models. In particular, both Erdős-Rényi and scale-free graphs produce adjacency matrix spectra which are close to being symmetric about 0. The mean spectra in Figure 3, on the other hand, have a peak near -1, with much of the spectrum negative but with a long positive tail. This asymmetry is predicted by Corollary 7, but is quite striking in these plots when comparing them to the plots for scale-free and Erdős-Rényi graphs in [3]. For the larger values of r in Figure 4, the spectrum appears to be close to symmetric about -1.

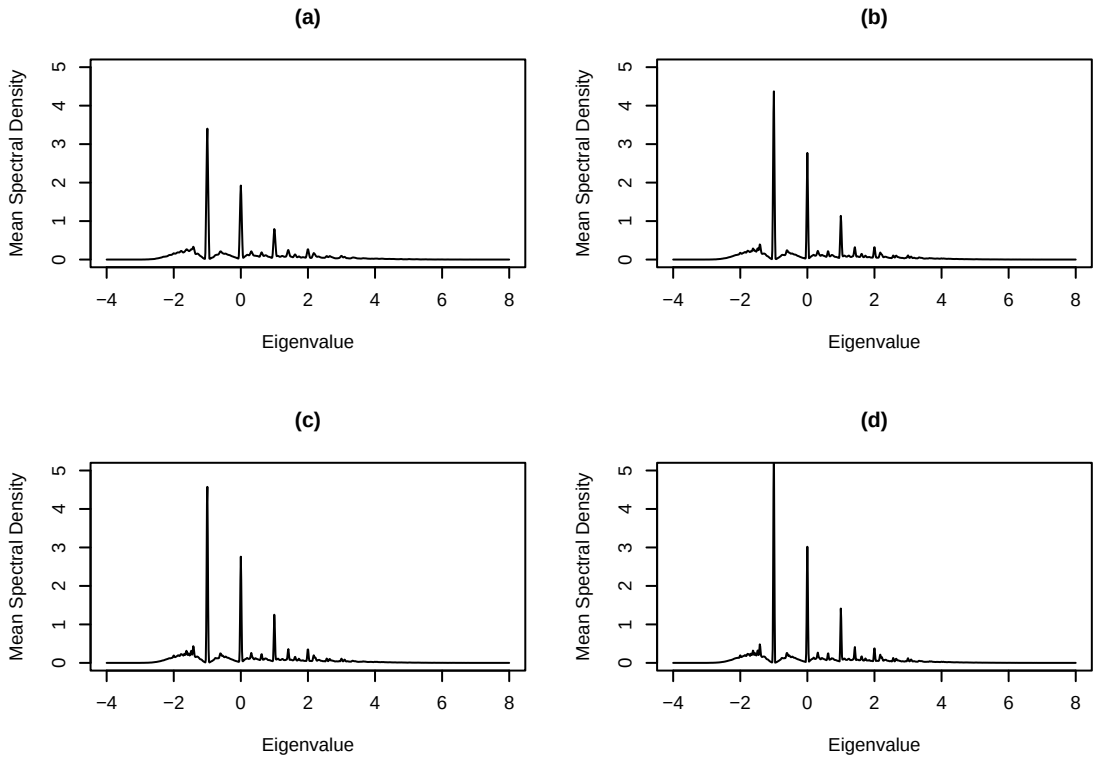

Figure 2: Simulation of spectra of random geometric graphs on the unit square with average degree 2.5. (a) 100 vertices, (b) 500 vertices, (c) 1000 vertices, (d) 2000 vertices. Each plot is an average of the spectral densities of 500 simulated graphs.

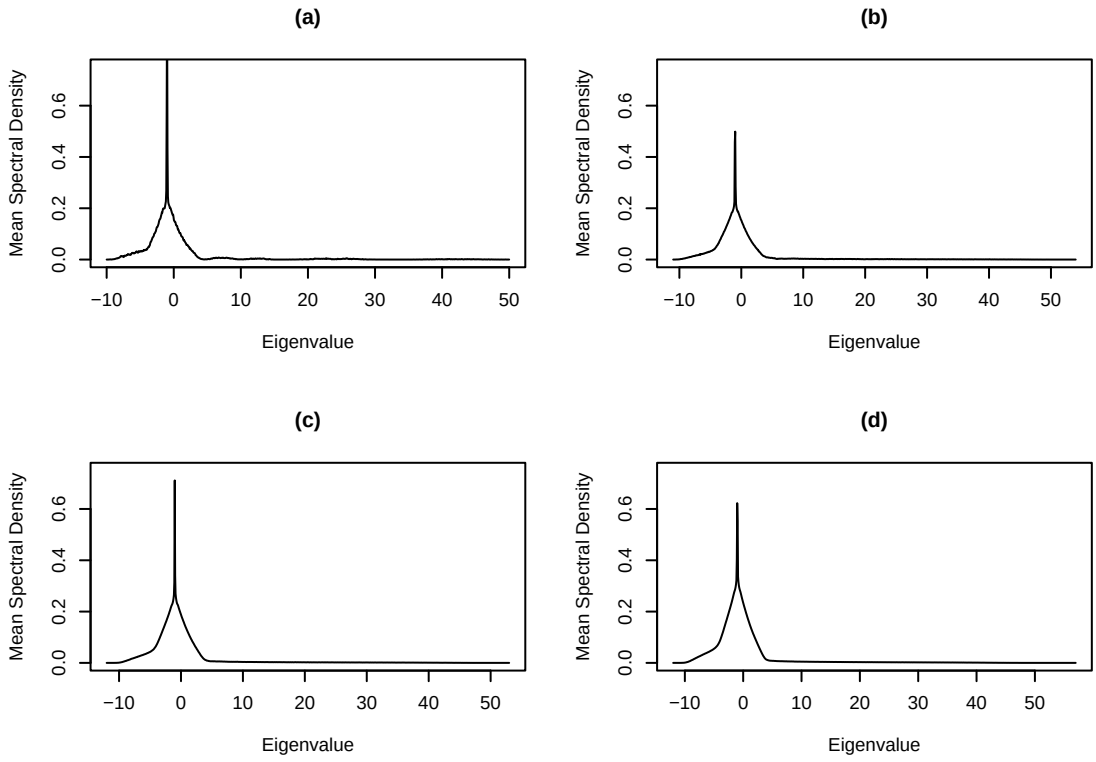

Figure 3: Simulation of spectra of random geometric graphs on the unit square with average degree 37.6. (a) 100 vertices, (b) 500 vertices, (c) 2000 vertices, (d) 5000 vertices. Each plot is an average of the spectral densities of 500 simulated graphs.

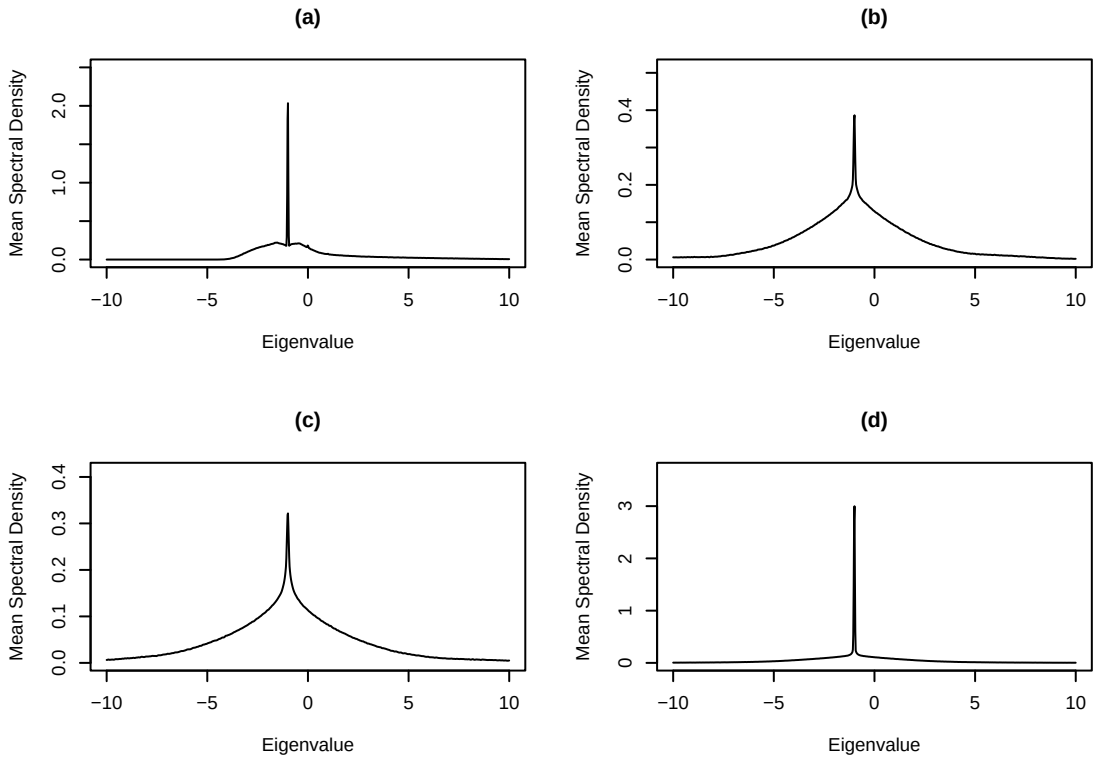

Figure 4: Simulation of spectra of random geometric graphs on the unit square with $n = 1000$ and different thresholds (a) $r = 0.05$, (b) $r = 0.2$, (c) $r = 0.4$, (d) $r = 0.8$. Each plot is an average of the spectral densities of 500 simulated graphs.

The simulations for small world graphs in [3] do show an asymmetry in the spectra and an associated high third moment of the spectral density. However they do not particularly resemble our plots for random geometric graphs; in particular they lack the singularities and the sharp peak in the spectral density apparent at -1 in our plots.

In [11], random geometric graphs are suggested as models for protein-protein interaction networks, based on analysis of "motifs" (small subgraphs which occur frequently in the graphs). We note that, in an analysis of the spectrum of a protein-protein interaction network in [7], the spectrum shows high multiplicities of both 0 and -1 eigenvalues.

References

[1] R. Albert, A.-L. Barabási, and H. Jeong. Mean-field theory for scale-free random networks. *Physica A*, 272:173–187, 1999.

[2] A.-L. Barabási, I. Derényi, I. Farkas, H. Jeong, Z. Néda, Z. N. Oltvai, E. Ravasz, A. Schubert, and T. Vicsek. Networks in life: scaling properties and eigenvalue spectra. *Physica A*, 314:25–34, 2002.

[3] A-L. Barabási, I. Derényi, I. J. Farkas, and T. Vicsek. Spectra of "real-world" graphs: Beyond the semi circle law. *Phys. Rev. E*, 64:026704, 2001.

[4] B. Bollobás. *Random Graphs.* Academic Press, London, 1985.

[5] F. R. K. Chung. *Spectral Graph Theory.* Number 92 in CBMS Regional Conference Series. AMS, Providence, Rhode Island, 1997.

[6] D. M. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs, Theory and Applications.* Academic Press, London, 1980.

[7] M. A. M. de Aguiar and Y. Bar-Yam. Spectral analysis and the dynamic response of complex networks. *Phys. Rev. E*, 71(016106), 2005.

[8] R. Durrett. *Probability: Theory and Examples.* Duxbury, 1996.

[9] O. Golinelli. Statistics of delta peaks in the spectral density of large random trees. http://arxiv.org/abs/cond-mat/0301437, 2003.

[10] M. Penrose. *Random Geometric Graphs.* Oxford University Press, 2003.

[11] N. Pržulj, D. G. Corneil, and I. Jurisica. Modeling interactome: scale-free or geometric. *Bioinformatics*, 20:3508–3515, 2004.

[12] R Development Core Team. *R: A language and environment for statistical computing.* R Foundation for Statistical Computing, Vienna, Austria, 2005. ISBN 3-900051-07-0.

[13] D. J. Watts and S. H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393:440–442, 1998.