A RANDOM HIERARCHICAL LATTICE: THE SERIES-PARALLEL
GRAPH AND ITS PROPERTIES

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Abstract
We consider a family of sequences of random graphs constructed by a hierarchical procedure. The construction depends on a parameter p. We investigate the effective resistance across the graphs, first-passage percolation on the graphs and the Cheeger constants of the graphs. In each case we find a phase transition at $p = \frac{1}{2}$.

Keywords: hierarchical lattice, effective resistance, phase transition, first-passage percolation, Cheeger constants

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1. Introduction

We examine the properties of a specific example from a broad class of graphs constructed by a random hierarchical procedure. The class of hierarchical lattices (we take the term lattice from the physics literature where it is equivalent to graph) was introduced in the physics literature as a simplified structure for investigating various statistical mechanical problems, [4], [5], [6], [7]. They provide approximations to the standard d-dimensional lattice but with a much simpler connection structure, which sometimes allows exact calculations to be done. Examples of such deterministic hierarchical constructions include the diamond lattice and pre-fractal graphs such as those associated with the Sierpiński gasket and other nested fractals, [2].

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The properties of hierarchical lattices can typically be described mathematically by a dynamical system on probability measures. We call this a random hierarchical system. Let \( \{X_j^{(0)}\}_{j \in \mathbb{N}} \) be independent identically distributed random variables and define \( X_j^{(n)} \) as

\[
f(X_{k(j-1)+1}^{(n-1)}, X_{k(j-1)+2}^{(n-1)}, \ldots, X_{kj}^{(n-1)})
\]

for some (non-linear) function \( f : \mathbb{R}^k \rightarrow \mathbb{R} \). Results about such iterations are obtained in \([8, 10, 13, 14, 15]\). In this setting it has been shown that there is a strong law of large numbers provided the function \( f \) satisfies certain conditions.

We wish to extend this to the case where the hierarchical lattice is itself random. The hierarchical systems which describe the properties of such lattices could be called random hierarchical systems in random environment. These lattices provide models for disordered lattices, a more complex setting for doing statistical mechanics. This type of random environment on regular hierarchical lattices is a natural setting for some computer science applications such as the biased coin problem \([1]\). We will regard the input variables \( \{X_j^{(0)}\}_{j \in \mathbb{N}} \) as either random or deterministic and then, to generate the random environment for our random hierarchical system, we choose the function \( f \) to be applied to the random variables randomly. We will obtain distributional fixed points for some such iterations.

As the range of possible lattices and systems generated in this way is vast we restrict ourselves to a particular random hierarchical lattice model which exhibits a range of random hierarchical systems. The specific model will be called the series-parallel graph.

1.1. The construction of the random hierarchical lattice

We set a parameter \( p \in [0, 1] \) and define a sequence of graphs \( G^{(n)} \). The first graph \( G^{(0)} \) consists of two vertices connected by a single edge, with the boundary of \( G^{(0)} \) considered to consist of these two vertices.

To construct \( G^{(n)} \) from \( G^{(n-1)} \), we take \( 2^n - 1 \) independent uniform \([0, 1]\) random variables \( \{Z_i^{(n-1)}\} \). We label the edges of \( G^{(n-1)} \) as \( 1, \ldots, 2^n - 1 \).

Then, if \( Z_i^{(n-1)} \leq p \), we replace edge \( i \) by a two-edge path i.e. we remove the original edge, add a new vertex, and add edges connecting the new vertex to the endpoints of the original edge.

If \( Z_i^{(n-1)} > p \), then we simply add a new edge between the same two vertices as
edge $i$. This means that the graph can have multiple edges.

The two boundary vertices of $G^{(n)}$ are inherited from $G^{(n-1)}$.

A typical graph at stage 5 with $p = \frac{1}{2}$ might look like that in Figure 1.

![Figure 1: An sample graph, $G^{(5)}$](image)

We will be concerned with the limiting properties of this sequence of random graphs.

1.2. Properties of the random hierarchical lattice

The first property we will consider is the total effective resistance between the two boundary points of the graph. In the random hierarchical system the function $f : \mathbb{R}^2 \to \mathbb{R}$ is chosen to be

$$f_1(x_1, x_2) = x_1 + x_2 \text{ with probability } p$$

$$f_2(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1} \text{ with probability } 1 - p.$$

There is clearly a phase transition in that for $p < \frac{1}{2}$ we see a predominance of parallel edges, while for $p > \frac{1}{2}$ the graphs are closer to a linear graph with some parallel edges. We formalise this in terms of the resistance. For $p < \frac{1}{2}$ we show that $X^{(n)}_1 \to 0$ in distribution, and for $p > \frac{1}{2}$ we show that $X^{(n)}_1 \to \infty$ in distribution. In the $p = \frac{1}{2}$ case we conjecture that there is no non-degenerate distributional fixed point for the hierarchical system. We also believe that there is an almost sure exponential growth rate for the resistance but can only establish upper and lower bounds on $\log(X^n_1)/n$ here.

In section 3, we investigate the limiting behaviour of the graph distance between the
two boundary points and first passage percolation on the graphs, which are described by a random hierarchical system which has a non-degenerate fixed point for \( p < \frac{1}{2} \).

Finally, in section 4, we study the behaviour of the Cheeger constants of the graphs, which gives an estimate for the spectral gap and hence the rate of convergence to equilibrium for a random walk on the graph. This also shows the phase transition at \( p = \frac{1}{2} \).

2. Resistance

We now put a resistance of 1 on each edge and define \( R^{(n)} \) to be the resistance between the boundary points in the graph \( G^{(n)} \).

We can define the following processes:

**Definition 2.1.** Set \( C^{(n)} \) to be the number of individual edges whose removal would disconnect \( G^{(n)} \). This behaves as a Galton-Watson branching process with each individual having 2 offspring with probability \( p \) and 0 offspring with probability \( 1 - p \).

Similarly set \( \tilde{C}^{(n)} \) to be the number \( w^{(n)}(1, 2) \) of edges connecting the two boundary points of \( G^{(n)} \). This also behaves as a branching process, with each individual having 2 offspring with probability \( 1 - p \) and 0 offspring with probability \( p \).

**Lemma 2.1.** In the case where \( p > \frac{1}{2} \), there is a positive probability of \( R^{(n)} \rightarrow \infty \).

**Proof.** The branching process \( C^{(n)} \) is a lower bound for the resistance \( R^{(n)} \) (as each edge has resistance 1) and its offspring mean is \( 2p + 0(1 - p) \), so it is supercritical for \( p > \frac{1}{2} \).

This also shows a positive probability that \( R^{(n)} > C(2p)^n \), for all \( n \), for some \( C > 0 \).

The offspring mean of \( (\tilde{C}^{(n)})_{n \in \mathbb{N}} \) is \( 2(1 - p) \), so it is supercritical for \( p < \frac{1}{2} \).

For \( n > 1 \), the graph \( G^{(n)} \) consists of 2 independent graphs \( G^{(n-1)}_1, G^{(n-1)}_2 \) (where \( G^{(n-1)}_j \) has resistance \( R^{(n-1)}_j \)), each equal in distribution to \( G^{(n-1)} \) connected together in a manner determined independently by \( G^{(1)} \).

Consider an event \( A \) determined by the sequence \( (G^{(n)})_{n \in \mathbb{N}} \) such that if \( G^{(1)} \) has a series connection then \( A = A_1 \cup A_2 \) (where \( A_1 \) and \( A_2 \) are the equivalent events for the sequences \( (G^{(n)}_1)_{n \in \mathbb{N}} \) and \( (G^{(n)}_2)_{n \in \mathbb{N}} \)), and that if \( G^{(1)} \) has a parallel connection then \( A = A_1 \cap A_2 \).
Lemma 2.2. If $p \neq \frac{1}{2}$, then any such event $A$ must have probability 0 or 1. This also applies if $\cap$ and $\cup$ are reversed in the above.

Proof. Let $q = \mathbb{P}(A)$. Then $q$ satisfies

$$q = h(q) = p(1 - (1 - q)^2) + (1 - p)q^2 = (1 - 2p)q^2 + 2pq$$

which has roots 0 and 1 unless $p = \frac{1}{2}$ in which case it reduces to $q = q$.

The second statement follows by the same method.

Corollary 2.1. We have $R_1^{(n)} \to \infty$ almost surely for $p > \frac{1}{2}$, and furthermore $2p$ as a lower bound on the exponential growth rate.

Proof. Both the events $\{R^{(n)} \to \infty\}$ and $\{\liminf_{n \to \infty} (R^{(n)})^{\frac{1}{n}} \geq 2p\}$ satisfy the conditions of Lemma 2.2. As we have already shown that they have positive probability, they must have probability 1.

Similarly we get

Corollary 2.2. We have $R^{(n)} \to 0$ almost surely for $p < \frac{1}{2}$, with exponential decay.

However the branching process argument does not help at $p = \frac{1}{2}$.

A polynomial $h(q)$ corresponding to $(1 - 2p)q^2 + 2pq$ can be constructed for sequences of graphs derived from similar constructions. In the case where there is just one function (e.g. a deterministic fractal lattice) then the polynomial $h(q)$ is studied in [11, 12], where it is shown that it has at most one fixed point in $(0, 1)$ which will be repulsive. It is also studied in [14]. These results do not apply to randomised constructions, but the fixed points of $h(q)$ still provide useful information.

We now consider the limiting behaviour of the resistance.

Lemma 2.3. The distribution of $R^{(n)}$ with parameter $p$ is the same as the distribution of $1/R^{(n)}$ with parameter $1 - p$.

Proof. Recall that the graph $G^{(n)}$ consists of 2 independent graphs $G_1^{(n-1)}, G_2^{(n-1)}$, with the same distribution as $G^{(n-1)}$ connected together in a manner determined independently by $G^{(1)}$.

Given $R^{(n)}$ we construct $\tilde{R}^{(n)}$ as follows:

We set $\tilde{R}^{(0)} = R^{(0)} = 1$. 
We define $\tilde{R}_1^{(n-1)}$ and $\tilde{R}_2^{(n-1)}$ to be independent random variables with the same distribution as $\tilde{R}^{(n-1)}$.

If $R^{(n)} = R_1^{(n-1)} + R_2^{(n-1)}$ then $\tilde{R}^{(n)} = \frac{\tilde{R}_1^{(n-1)} \tilde{R}_2^{(n-1)}}{\tilde{R}_1^{(n-1)} + \tilde{R}_2^{(n-1)}}$. 

If $R^{(n)} = \frac{R_1^{(n-1)} R_2^{(n-1)}}{R_1^{(n-1)} + R_2^{(n-1)}}$ then $\tilde{R}^{(n)} = \tilde{R}_1^{(n-1)} + \tilde{R}_2^{(n-1)}$.

Then $\tilde{R}^{(n)}$ with parameter $p$ is distributed as $R^{(n)}$ with parameter $1 - p$.

But $\tilde{R}^{(0)} = \frac{1}{R^{(0)}}$. Assuming (induction hypothesis), that $\tilde{R}^{(n-1)}$ has the same distribution as $\frac{1}{R^{(n-1)}}$ then if $R^{(n)} = R_1^{(n-1)} + R_2^{(n-1)}$ then 

$$\tilde{R}^{(n)} = \frac{1}{\tilde{R}_1^{(n-1)} + \tilde{R}_2^{(n-1)}}$$

and if $R^{(n)} = \frac{R_1^{(n-1)} R_2^{(n-1)}}{R_1^{(n-1)} + R_2^{(n-1)}}$ then 

$$\tilde{R}^{(n)} = \tilde{R}_1^{(n-1)} + \tilde{R}_2^{(n-1)}$$

So $\tilde{R}^{(n)} = \frac{1}{R^{(n)}}$ for all $k$

Corollary 2.2 showed that $R^{(n)} \to 0$ almost surely for $p < \frac{1}{2}$. In section 4, it will be useful to have a stronger result, namely that for any sequence of random variables $(X^{(n)})_{n \in \mathbb{N}}$ with $X^{(n)} \to \frac{1}{d} R^{(n)}$ for each $n$, $X^{(n)} \to 0$ almost surely (for $p < \frac{1}{2}$).

We can use recurrences for expectation to prove this. For $p < 1/3$, we can use exponential decay of expectation and apply a Borel-Cantelli Lemma. To extend this to the interval $[\frac{1}{3}, \frac{1}{2}]$ we take an integer $\alpha$ and consider the behaviour of $\sqrt[\alpha]{R^{(n)}}$.

Now

$$\mathbb{E}(\sqrt[\alpha]{R^{(n)}} | F_{n-1}) = p(R_1^{(n-1)} + R_2^{(n-1)})^{\frac{1}{\alpha}} + (1 - p) \left( \frac{R_1^{(n-1)} R_2^{(n-1)}}{R_1^{(n-1)} + R_2^{(n-1)}} \right)^{\frac{1}{\alpha}}.$$

So if

$$p(1 + x)^{\frac{1}{\alpha}} + (1 - p) \left( \frac{x}{1 + x} \right)^{\frac{1}{\alpha}} \leq \frac{C}{2} (1 + x^{\frac{1}{\alpha}})$$
for some $C$ (depending on $\alpha$) and for all $x > 0$, then
\[
\mathbb{E} \sqrt{nR(n)} \leq C^n,
\]
so if $C < 1$ then $\sqrt[n]{X(n)} \to 0$ almost surely, if $X(n) =_d R(n)$ for all $n$, using a Borel-Cantelli Lemma again.

Hence we are interested in the supremum over $x > 0$ of
\[
f(x, \beta, p) = \frac{p(1 + x)^{\beta} + (1 - p) \left( \frac{x}{1 + x} \right)^{\beta}}{1 + x^{\beta}}
\]
where $\beta = \frac{1}{\alpha}$.

**Lemma 2.4.** When $p \leq \frac{1}{2}$ and $\alpha \in \mathbb{N}$, the supremum of $f(x, \beta, p)$ over $x > 0$ is attained at $x = 1$.

**Proof.** Differentiate $f(x, \beta, p)$ to get
\[
\frac{\partial f(x, \beta, p)}{\partial x} = \frac{\beta}{(1 + x^{\beta})^2} \left[ (1 + x^{\beta})(p(1 + x)^{\beta-1} + (1 - p) \left( \frac{x}{1 + x} \right)^{\beta-1} \left( \frac{1}{1 + x} \right)^2 \right)
\]
\[
- x^{\beta-1} \left( p(1 + x)^{\beta} + (1 - p) \left( \frac{x}{1 + x} \right)^{\beta} \right)
\]
\[
= \frac{\beta}{(1 + x^{\beta})^2 (1 + x)^{\beta-1}} \left[ p(1 + x)^{2\beta} (1 - x^{\beta-1}) + (1 - p)x^{\beta-1}(1 - x^2) \right].
\]
For $x \leq 1$, $(1 - x^{\beta-1}) \leq 0$ (because $\beta \leq 1$) and similarly $(1 - x^2) \geq 0$. The opposite is the case for $x \geq 1$. So if we define
\[
g(x, \beta) = (1 + x)^{2\beta} (1 - x^{\beta-1}) + x^{\beta-1}(1 - x^2)
\]
then for $p \leq \frac{1}{2}$ we have for $0 < x \leq 1$
\[
\frac{\partial f(x, \beta, p)}{\partial x} \geq \frac{\beta}{2(1 + x^{\beta})^2 (1 + x)^{\beta+1}} g(x, \beta)
\]
and for $x \geq 1$ we have
\[
\frac{\partial f(x, \beta, p)}{\partial x} \leq \frac{\beta}{2(1 + x^{\beta})^2 (1 + x)^{\beta+1}} g(x, \beta).
\]
Hence the sign of \( \frac{\partial f(x, \beta, p)}{\partial x} \) is the same as that of \( g(x, \beta) \) for all \( x > 0 \). Now

\[
g(x, \beta) = x^{\beta - 1} \left[ (1 - x^2) - (1 + x)^{2\beta}(1 - x^{1-\beta}) \right]
\]

and since \( 2 = (2\alpha)\beta \) and \( 1 - \beta = (\alpha - 1)\beta \) and \( 2\alpha \) and \( \alpha - 1 \) are both integers, we can take a factor of \( (1 - x^{1-\beta}) \) out as follows:

\[
g(x, \beta) = x^{\beta - 1}(1 - x^2)^{(1 + x + x^{2\beta} + \ldots + x^{(2\alpha - 1)\beta})} - (1 + x)^{2\beta}(1 + x^{2\beta} + \ldots + x^{(\alpha - 2)\beta})
\]

So when \( \alpha \geq 2 \) (and so \( \beta \leq \frac{1}{2} \)),

\[
g(x, \beta) > 0 \text{ when } 0 < x < 1
\]

\[
< 0 \text{ when } x > 1.
\]

So \( f(x, \beta, p) \) is maximised over \( x > 0 \) by \( f(1, \beta, p) \).

Also \( f(x, 1, p) = p + (1 - p)\frac{x}{(1+x)^2} \), which is also maximised over \( x > 0 \) at \( x = 1 \).

**Corollary 2.3.** For all \( p < \frac{1}{2} \), if \( X^{(n)} = d \ R^{(n)} \) for all \( n \), we have \( X_1^{(n)} \to 0 \), almost surely.

**Proof.** Fix \( \alpha \geq 2 \). Use Lemma 2.4 to get that \( \sqrt{X^{(n)}} \) converges almost surely to zero if \( f(1, 1/\alpha, p) < \frac{1}{2} \). Now

\[
2f(1, 1/\alpha, p) = p(\sqrt{2}) + (1 - p) \left( \frac{1}{\sqrt{2}} \right)
\]

so we need

\[
(2^{\frac{1}{2}} - 1)p < 2^{1-\alpha} - 1
\]

which gives

\[
p < \frac{1}{2^{\frac{1}{2}} + 1}.
\]

But

\[
\lim_{\alpha \to \infty} \frac{1}{2^{\frac{1}{2}} + 1} = \frac{1}{2}
\]

so the result follows.

**Corollary 2.4.** For all \( p > \frac{1}{2} \), if \( X^{(n)} = d \ R^{(n)} \) for all \( n \), we have \( X_1^{(n)} \to \infty \), almost surely.

**Proof.** This follows immediately from Lemma 2.3 and Corollary 2.3.
To get some more information about growth or decay rates, we investigate the behaviour of $\mu^{-n}R^{(n)}$. We define $f(x, \beta, p)$ as for Lemma 2.4 and define $\epsilon(\alpha)$ such that
\[
\frac{1}{2} + \epsilon(\alpha) = \sup_{p} \{ f(1, \beta, p) \geq f(x, \beta, p) \text{ for all } x \geq 0 \}
\]
where $\beta = 1/\alpha$. As $f$ is continuous the supremum will be attained.

**Lemma 2.5.** If $p < \frac{1}{2} + \epsilon(\alpha)$ and
\[
\mu > (2f(1, \frac{1}{\alpha}, p))^\alpha
\]
for some $\alpha \in \mathbb{N}$ then
\[
\mu^{-n}R^{(n)} \to 0
\]
almost surely and furthermore this applies to any sequence of random variables with these distributions.

**Proof.** This follows from the fact that
\[
\mathbb{E} \left( \sqrt[n]{\mu^{-n}R^{(n)}} \right) \leq \left( 2\mu^{-\frac{1}{\alpha}} f(1, \frac{1}{\alpha}, p) \right)^n
\]
via the usual argument involving the Borel-Cantelli Lemma.

We note that Lemma 2.4 implies that $\epsilon(\alpha) \geq 0$ for $\alpha \in \mathbb{N}$. This is enough to give the following:

**Corollary 2.5.** When $p = \frac{1}{2}$ and $X^{(n)} = d R^{(n)}$ for all $n$, \[
\frac{1}{n} \log X^{(n)} \to 0 \text{ a.s. as } n \to \infty.
\]

**Proof.** Take $\mu > 1$. We require, for some $\alpha \in \mathbb{N}$,
\[
\frac{1}{2} \left( \sqrt{2} + \sqrt{\frac{1}{2}} \right) < \mu^\frac{1}{n}
\]
but
\[
\lim_{\alpha \to \infty} \left( \frac{1}{2} \left( \sqrt{2} + \sqrt{\frac{1}{2}} \right) \right)^\alpha = 1,
\]
so for sufficiently large $\alpha$ we will have $\mu^{-n}X^{(n)} \to 0$ almost surely, from Lemma 2.5.

We use the symmetry properties to obtain, for $\mu < 1$, $\mu^{-n}X^{(n)} \to \infty$ a.s., which gives the result.
Lemma 2.6. We have that
\[ \epsilon(\alpha) > 0 \text{ for all } \alpha \in \mathbb{N}. \]

Proof. We know that \( \epsilon(1) = \frac{1}{2} \). From the proof of Lemma 2.4 we see that the function \( \hat{f}(x, \beta, p) = f(x, \beta, p) - f(1, \beta, p) \) is strictly negative when \( p = \frac{1}{2}, \beta = 1/\alpha, \alpha \in \mathbb{N} \), except when \( x = 1 \). As this \( \hat{f}(x, \beta, p) \) is infinitely differentiable in \( x \) and \( p \) (if \( x > 0 \)), and \( \hat{f}(1, \beta, p) = 0 \) and \( \frac{\partial \hat{f}}{\partial x} = 0 \) for all \( \beta, p \), to show that, for each \( \alpha \in \mathbb{N} \), \( \hat{f}(x, 1/\alpha, p) \leq 0 \) for all \( x \), for some \( p > \frac{1}{2} \), it remains to show that
\[ \frac{\partial^2 \hat{f}}{\partial^2 x}(1, \beta, \frac{1}{2}) < 0, \]
so that this second derivative will remain negative for \( p \) in a neighbourhood of \( \frac{1}{2} \).

Now we can calculate
\[ \frac{\partial^2 \hat{f}}{\partial^2 x}(1, \beta, \frac{1}{2}) = \frac{\beta}{2^{2\beta + 4}}(2^{2\beta}(1 - \beta) - \beta - 1). \]

We know \( \frac{\partial^2 \hat{f}}{\partial^2 x}(1, \beta, \frac{1}{2}) \leq 0 \), so we have the result as long as there are no solutions of
\[ \beta = \frac{2^{2\beta} - 1}{2^{2\beta} + 1} \]
in \((0, 1)\). But
\[ \frac{d}{d\beta} \left( \frac{2^{2\beta} - 1}{2^{2\beta} + 1} \right) = \frac{2^{2\beta + 2} \log 2}{(2^{2\beta} + 1)^2} \]
and as \( \frac{4y}{(y+1)^2} < 1 \) for \( y > 1 \) the derivative is bounded above by \( \log 2 \) in \((0, 1)\), giving the result.

Corollary 2.6. If \( X^{(n)} = d R^{(n)} \) for all \( n \), then
a) For \( p > \frac{1}{2} \), there exists a value \( \lambda(p) > 0 \) such that, almost surely,
\[ (2p - 1) \log 2 \leq \frac{1}{n} \log X^{(n)} \leq \lambda(p) \text{ eventually}. \]

Furthermore, \( \lambda(p) \to 0 \) as \( p \downarrow \frac{1}{2} \).

b) For \( p < \frac{1}{2} \),
\[ (2p - 1) \log 2 \geq \frac{1}{n} \log X^{(n)} \geq -\lambda(1 - p) \text{ eventually}. \]
Proof. Upper bound of part a):
We use Lemma 2.5 to see that, if \( p < \frac{1}{2} + \epsilon(\alpha) \) and
\[
\mu > \left( p \sqrt{2} + (1 - p) \sqrt{\frac{1}{2}} \right)^\alpha
\]
then
\[
\mu^{-n} X^{(n)} \to 0 \text{ a.s. as } n \to \infty.
\]
So we define
\[
\lambda(p) = \inf_{p < \frac{1}{2} + \epsilon(\alpha)} \alpha \log \left( p \sqrt{2} + (1 - p) \sqrt{\frac{1}{2}} \right).
\]
Now we let \( p \downarrow \frac{1}{2} \) and use Lemma 2.6 to allow us to take the limit as \( \alpha \to \infty \). The lower bound of part b) follows by symmetry.

Upper bound of part b):
For all \( p < \frac{1}{2} \) and \( \mu > \left( p \sqrt{2} + (1 - p) \sqrt{\frac{1}{2}} \right)^\alpha \) we have \( \mu^{-n} X^{(n)} \to 0 \text{ a.s. as } n \to \infty \).
Take the limit of \( \mu \) as \( \alpha \to \infty \), using l'Hôpital's rule, to get the result.

The lower bound of part a) follows by symmetry.

We remark that a more detailed calculation, given in [9], shows that when \( p = 1/2 \) we have \( \mathbb{E}(\log X^{(n)})^2 \to \infty \) as \( n \to \infty \). We conjecture that the limiting probability measure in the critical case puts mass 1/2 at 0 and \( \infty \).

3. The length of the graph

3.1. Introduction

We consider the graph distance between the two boundary points. To study this, we label the vertices as follows: the initial vertices are 1 and 2, and new vertices are numbered in order of their insertion i.e. when an edge \( a \leftrightarrow b \) is replaced by two edges in series these are \( a \leftrightarrow c \) and \( c \leftrightarrow b \) with \( c > a, b \). We will define \( j(c) \) to be the stage at which vertex \( c \) was added to the graph i.e. \( j(c) = \min_n \{ c \in V(G^{(n)}) \} \) and further define \( n(c) \) to be a neighbour of \( c \) in \( G^{(j(c))} \) i.e. \( n(c) = \min_n \{ a \leftrightarrow c \text{ in } G^{(j(c))} \} \). We take \( n(1) = 2 \) and \( n(2) = 1 \), and \( j(1) = j(2) = 0 \).

We define the distance \( d^{(n)}(a, b) \) to be the graph distance in \( G^{(n)} \) between \( a \) and \( b \), for \( a, b \in V(G^{(n)}) \).
We now define $K^{(n)} = d^{(n)}(1, 2)$, the distance between the two boundary points. This can be thought of as the ‘length’ of the graph.

In the hierarchical system framework, $f$ is again chosen randomly from the two functions

$$
f_1(x_1, x_2) = x_1 + x_2 \text{ with probability } p
$$

$$
f_2(x_1, x_2) = \min(x_1, x_2) \text{ with probability } 1 - p.
$$

We define $\kappa_p$ to be the map on probability measures $\nu$ on $\mathbb{R}^+$ associated with this random hierarchical system. That is, given a probability measure $\nu$ on $\mathbb{R}^+$, we can take two independent random variables $X_1$ and $X_2$ with law $\nu$, and an independent uniform $[0, 1]$ random variable $Z$, and define a random variable

$$
X = \begin{cases} 
X_1 + X_2 & Z \leq p \\
\min(X_1, X_2) & Z > p 
\end{cases}
$$

setting $\kappa_p(\nu)$ to be the law of $X$.

Then, if the random variables $K^{(0)}$ have law $\nu^{(0)}$, the random variables $K^{(n)}$ have law $\kappa_p^n(\nu^{(0)})$.

**Lemma 3.1.** a) If $p > \frac{1}{2}$ then $K^{(n)}$ grows exponentially quickly with probability 1.

b) If $p < \frac{1}{2}$ then $K^{(n)}$ has a finite limit $K$ as $n \to \infty$, with probability 1.

**Proof.** Recall the branching processes defined in Definition 2.1.

For a), we use the branching process $C^{(n)}$ from Definition 2.1 to show that the probability is positive, because $K^{(n)} \geq C^{(n)}$, and then use Lemma 2.2 on the event $K^{(n)} \to \infty$ to see that it must be 1.

For b), we use the other branching process $\tilde{C}^{(n)}$ which is the number of edges $1 \leftrightarrow 2$. This is supercritical when $p < \frac{1}{2}$, so with positive probability we have $K^{(n)} = 1$ for all $n$. Now we use Lemma 2.2 again on the same event to get the result.

**Corollary 3.1.** When $p < \frac{1}{2}$, the map $\kappa_p$ has a non-degenerate fixed point.

**Proof.** The law of the random variable $K$ gives such a fixed point.

### 3.2. The fixed points of $\kappa_p$

We now consider the iteration of $\kappa_p$. For $p < \frac{1}{2}$, one fixed point is the distribution $\nu_p$ of the random variable $K$ found in Lemma 3.1. We will see that for this range of $p$
this fixed point is essentially unique while for \( p > \frac{1}{2} \) there are no non-degenerate fixed points for distributions on \( \mathbb{R}^+ \).

We consider labelling the edges of the graph \( G^{(n)} \) with i.i.d. random variables on \( \mathbb{R}^+ \) with distribution \( \mu^{(0)} \). We let \( L^{(n)} \) be the minimum sum of the labels on a route between the endpoint vertices. This describes first-passage percolation on the graph \( G^{(n)} \).

Also let \( \mu^{(n)} = \kappa_n^p(\mu^{(0)}) \).

**Lemma 3.2.** The distribution of \( L^{(n)} \) is \( \mu^{(n)} \).

**Proof.** The statement is obvious for \( n = 0 \). For larger \( n \) we assume the statement for \( n-1 \) and note that \( G^{(n)} \) consists of two i.i.d. graphs with the distribution of \( G^{(n-1)} \) connected in series with probability \( p \) and in parallel with probability \( 1 - p \). In the former case \( L^{(n)} \) is the sum of two independent variables with distribution \( \mu^{(n-1)} \) and in the latter case it is the minimum of these variables. This describes the map \( \kappa_p \) so proves the lemma by induction.

We now investigate the sequence \( (L^{(n)})_{n \in \mathbb{N}} \). We define \( \lambda \) to be the infimum of the support of \( \mu^{(0)} \) i.e. \( \lambda = \inf \{x : \mu^{(n)}(-\infty, x) > 0\} \). Note that the law \( \mathcal{L}(\lambda K) \) of \( \lambda K \) is a fixed point of \( \kappa_p \) for \( p < \frac{1}{2} \).

**Theorem 3.1.** For \( p < \frac{1}{2} \), as \( n \to \infty \),

\[
\kappa_n^p(\mu^{(n)}) \to \mathcal{L}(\lambda K) \text{ (weak convergence)}.
\]

**Proof.** We condition on the (replacement) sequence of graphs \( G^{(n)} \). We know from Lemma 3.1 that for \( n \) large enough the ‘length’ \( K^{(n)} = K \). First consider the case where \( K = 1 \). In this case the branching process \( \hat{C}^{(n)} \) from Definition 2.1 grows exponentially almost surely. The value \( L^{(n)} \) is bounded above by the minimum label on these \( \hat{C}^{(n)} \) edges. But this minimum value converges weakly to \( \lambda \).

Similarly when \( K = k \), we consider \( n_0 \) large enough that \( d^{(n_0)}(1, 2) = k \). We consider the paths of length \( k \) between vertices 1 and 2 in \( G^{(n_0)} \), and note that \( K = k \) implies that one such path must be preserved in \( G^{(n)} \) for \( n > n_0 \). Now the self-similarity of the structure shows that each edge in this path has a branching process associated with it, with the same offspring distribution as \( \hat{C}^{(n)} \), and also growing exponentially.
conditioned on survival. Hence the number of edge-disjoint paths from 1 to 2 of length $k$ grows exponentially, and hence the weak convergence result holds.

This shows that $\mathbb{P}(L^{(n)} > \lambda K) \to 0$ as $n \to \infty$. Finally it is obvious that $\mathbb{P}(L^{(n)} < \lambda K^{(n)}) = 0$, completing the proof.

This shows that these are the only fixed points and that starting with any distribution on $\mathbb{R}^+$ the sequence will converge to one of them.

Note that if $\lambda = 0$ the limit is a point mass at 0.

For $p \geq \frac{1}{2}$ the lower bound in the proof shows that if $\lambda > 0$ then $L^{(n)} \to \infty$ because in that case $K^{(n)} \to \infty$. Furthermore if $p > \frac{1}{2}$ then $K^{(n)}$ grows exponentially fast almost surely, which is enough to show that the only fixed point is a point mass at 0.

In the $p < \frac{1}{2}$ case this shows that in the limit first-passage percolation on the graphs $G^{(n)}$ loses all randomness except that coming from the random structure of the graphs.

4. Cheeger constants

For a subset $S \subseteq G^{(n)}$, following [3], we define $\text{vol} S$ to be the sum of the degrees of vertices in $S$, and define $\bar{S}$ to be the complement of $S$ in $V(G^{(n)})$. Further we define $E(S_1, S_2)$ to be the number of edges between vertices in $S_1$ and vertices in $S_2$.

We now define

$$h_{G^{(n)}}(S) = \frac{|E(S, \bar{S})|}{\min\{\text{vol} S, \text{vol} \bar{S}\}}$$

and the Cheeger constant

$$h_{G^{(n)}} = \min_S h_{G^{(n)}}(S).$$

This is a measure of the connectivity of the graph, see [3] for further details. Where $\lambda_1(G^{(n)})$ is the smallest positive eigenvalue of $G^{(n)}$ as defined in [3], we have the Cheeger inequality (proved in [3]):

$$2h_{G^{(n)}} \geq \lambda_1(G^{(n)}) > \frac{h_{G^{(n)}}^2}{2}.$$

So bounds on the Cheeger constant can give bounds on the lowest positive eigenvalue.

We consider the asymptotics of the Cheeger constants of the series-parallel graphs $G^{(n)}$.

**Lemma 4.1.** The Cheeger constant satisfies

$$\frac{1}{n} \log h_{G^{(n)}} \geq \log \frac{1}{2}. $$
Furthermore, if \( p > \frac{1}{2} \), this bound is asymptotically tight, i.e.

\[
\lim_{n \to \infty} \frac{1}{n} \log h_{G^{(n)}} = \log \frac{1}{2}.
\]

**Proof.** As \( G^{(n)} \) is connected, \(|E(S, \bar{S})| \geq 1\) for any subset \( S \subseteq V(G^{(n)})\). Furthermore, \( \text{vol} G^{(n)} = 2^{n+1} \) as there are \( 2^n \) edges in \( G^{(n)} \). So \( \min\{\text{vol} S, \text{vol} \bar{S}\} \leq 2^n \), giving the lower bound.

To prove that it is tight, we consider the branching process \( C^{(n)} \) from Definition 2.1, which is the number of 1-cuts (i.e. edges whose removal would disconnect the graph). This is supercritical when \( p > \frac{1}{2} \) so with positive probability \( C^{(n)} \to \infty \). So if we have a vertex set \( S^{(n)}(n) \subseteq V(G^{(n)}) \), there is positive probability of the existence of descendant sets \( S^{(m)} \subseteq V(G^{(m)}) \), \( m > n \), such that \( S^{(n)} \subseteq S^{(m)} \), \( \bar{S}^{(n)} \subseteq \bar{S}^{(m)} \) and \( |\partial S^{(m)}| = |\partial S^{(n)}| \) for all \( m \). If \( S^{(n)} \) and \( \bar{S}^{(n)} \) both contain an interior edge then \( \text{vol} S^{(m)} \) and \( \text{vol} \bar{S}^{(m)} \) will both be at least \( \epsilon 2^m \) for some \( \epsilon > 0 \) so this is enough to show that there is positive probability that \( \lim_{n \to \infty} \frac{1}{n} \log h_{G^{(n)}} = \log \frac{1}{2} \). Now note that the existence of a set \( S^{(n)} \) (of any one of the sets \( G^{(n)} \)) with a sequence of \( S^{(m)}, m > n \) satisfying the above property is an event of the same type as convergence of \( R^{(n)} \) to \( \infty \); so by the proof of Lemma 2.2 its probability must be 0 or 1. This gives the result.

The intuition here is that when \( p > \frac{1}{2} \) the structure looks like a relatively small number of long strands, so it is easy to make small cuts. We do not expect this to be true for small \( p \) where the graph should be more connected.

The following gives an upper bound on the Cheeger constant when \( p \leq \frac{1}{2} \):

**Lemma 4.2.** The sequence of Cheeger constants satisfies

\[
\limsup_{n \to \infty} \frac{1}{n} \log h_{G^{(n)}} \leq \log \frac{2 - p}{2}
\]

almost surely.

**Proof.** Construct a sequence of sets of vertices \( S^{(n)}(n) \subseteq V(G^{(n)}); n \in \mathbb{N} \) as follows: At stage 0 put vertex 1 \( \in S^{(n)} \), vertex 2 \( \in \bar{S}^{(n)} \). Then at stage \( n + 1 \) consider each edge of \( G^{(n)} \) in \( E(S^{(n)}, \bar{S}^{(n)}) \). The edge will be \( a \leftrightarrow b \) with \( a \in S^{(n)} \) and \( b \in \bar{S}^{(n)} \). If the edge is replaced by a pair of edges in parallel \( a \leftrightarrow b \) then keep \( a \in S^{(n+1)} \) and \( b \in \bar{S}^{(n+1)} \). If the edge is replaced with a pair of edges in series \( a \leftrightarrow c \) and \( c \leftrightarrow b \) then keep \( a \in S^{(n)} \) and \( b \in \bar{S}^{(n+1)} \), and place the new vertex \( c \in S^{(n)} \) with probability \( \frac{1}{2} \).
This ensures that \( \frac{1}{n} \log(\text{vol } S^{(n)}) \to \log 2 \) and \( \frac{1}{n} \log(\text{vol } \bar{S}^{(n)}) \to \log 2 \), almost surely, and further that \( |E(S^{(n)}, \bar{S}^{(n)})| \) behaves as a branching process with offspring distribution 1 with probability \( p \), 2 with probability \( 1 - p \), and hence \( \frac{1}{n} \log |E(S^{(n)}, \bar{S}^{(n)})| \to \log(2 - p) \) almost surely. This gives the result.

**Lemma 4.3.** The sequence of Cheeger constants satisfies

\[
\liminf_{n \to \infty} \frac{1}{n} \log h_{G^{(n)}} \geq -2p \log 2.
\]

*Proof.* We let \( S^{(n)} \) be a subset of the vertex set such that

\[
h_{G^{(n)}} = \frac{|E(S^{(n)}, \bar{S}^{(n)})|}{\text{vol } S^{(n)}}.
\]

Note that we may assume that \( \bar{S}^{(n)} \) forms a connected subgraph - this is because

\[
\frac{a + b}{c + d} \geq \min\left(\frac{a}{c}, \frac{b}{d}\right)
\]

for positive \( a, b, c, d \) so a disconnected \( \bar{S}^{(n)} \) will always contain a connected component giving at least as good a Cheeger constant.

We define \( k^{(n)} = \min\{a \in \bar{S}^{(n)}\} \) and \( \hat{k}^{(n)} \) to be its neighbour in \( G^{(j(k^{(n)}))} \), i.e. \( n(k^{(n)}) \). We consider the evolution of the edges replacing the edge \( \hat{k}^{(n)} \leftrightarrow k^{(n)} \) in \( G^{(j(k^{(n)}))} \). (Note that no vertex \( c \) with \( j(c) < j(k^{(n)}) \) can be in \( \bar{S}^{(n)} \).)

The resistance \( R^{(n)}(\hat{k}^{(n)}, k^{(n)}) \) between \( \hat{k}^{(n)} \) and \( k^{(n)} \) in \( G^{(n)} \) can be bounded below by the reciprocal of the minimum cut between \( \hat{k}^{(n)} \) and \( k^{(n)} \), which is at least

\[
\frac{1}{|E(S^{(n)}, \bar{S}^{(n)})|}.
\]

So

\[
|E(S^{(n)}, \bar{S}^{(n)})| \geq \frac{1}{R^{(n)}(\hat{k}^{(n)}, k^{(n)})}.
\]

Because of the connectivity of \( \bar{S}^{(n)} \) and the graph structure, the volume satisfies

\[
\text{vol } \bar{S}^{(n)} \leq 2^{n - j(k^{(n)}) + 1}
\]

(any volume larger than this containing \( k^{(n)} \) will contain one of the two original (i.e. in \( G^{(j(k^{(n)}))} \)) neighbours of \( k^{(n)} \), which are not in \( \bar{S}^{(n)} \) by hypothesis) so

\[
h_{G^{(n)}} \geq \frac{1}{2^{n - j(k^{(n)}) + 1} R^{(n)}(\hat{k}^{(n)}, k^{(n)})}.
\]

Now

\[
R^{(n)}(\hat{k}^{(n)}, k^{(n)}) = d R^{(n - j(k^{(n))})}.
\]
So we use the complete convergence results of Lemma 2.5 and Corollary 2.6 as follows:

\[ h_{G(n)} \geq d \frac{1}{2n-j(k^{(n)})+1 R(n-j(k^{(n)}))} \]

\[ 2(2\mu)^{n-j(k^{(n)})} h_{G(n)} \geq d \frac{1}{\mu^{-(n-j(k^{(n)})+1) R(n-j(k^{(n)}))}}. \]

Using Corollary 2.6 we choose \( \mu > 2^{2p-1} \) so that

\[ \frac{1}{\mu^{-(n-j(k^{(n)})+1) R(n-j(k^{(n)}))}} \rightarrow \infty \text{ a.s.} \]

as long as \( n-j(k^{(n)}) \rightarrow \infty \) a.s. This condition is satisfied because of the bound above on the volume and the lim sup result for the Cheeger constant in Lemma 4.2.

So, for \( \mu > 2^{2p-1} \),

\[ (2\mu)^{n} h_{G(n)} \geq (2\mu)^{n-j(k^{(n)})} h_{G(n)} \rightarrow \infty \]

almost surely, because of the complete convergence in Lemma 2.5.

So \( \log 2\mu + \liminf_{n \rightarrow \infty} \frac{1}{n} \log h_{G(n)} \geq 0 \), which gives the result.

We now put Lemmas 4.1, 4.2 and 4.3 together to obtain the following:

**Theorem 4.1.** When \( p > \frac{1}{2} \),

\[ \lim_{n \rightarrow \infty} \frac{1}{n} \log h_{G(n)} = -\log 2 \]

almost surely.

When \( p \leq \frac{1}{2} \),

\[ \liminf_{n \rightarrow \infty} \frac{1}{n} \log h_{G(n)} \geq -2p \log 2 \]

and

\[ \limsup_{n \rightarrow \infty} \frac{1}{n} \log h_{G(n)} \leq \log \frac{2}{2} - p, \]

also almost surely.

This gives a clear transition in behaviour as \( p \) passes through \( \frac{1}{2} \).

**References**


