

Renormalisation of random hierarchical systems

Jonathan Jordan

1st August 2003

Contents

I	Introduction	5
1	Introduction and preliminaries	6
1.1	Random hierarchical systems	6
1.2	Random fractal graphs	10
II	Asymptotics for random hierarchical systems	19
2	Almost-sure convergence	20
2.1	Introduction	20
2.2	The randomised model	22
2.3	Required lemmas	31
3	Random fractal resistor networks	37
3.1	Introduction	37

3.1.1	A zero-one law	39
3.2	Asymptotics for the series-parallel network	41
3.2.1	Symmetry properties	41
3.2.2	Behaviour of $\log X_1^{(n)}$	42
3.2.3	Limiting behaviour for $p \neq \frac{1}{2}$	49
3.2.4	Another related hierarchical system	57
3.3	More examples of random fractal graphs	59
3.3.1	Introduction	59
3.3.2	A non-self-dual network	60
3.3.3	More rapid growth or decay	63
3.3.4	Another self-dual example	64
4	The length of the network	66
4.1	Introduction	66
4.2	The map κ_p	66
4.3	The fixed points of κ_p	69
5	Another system with a non-degenerate fixed point	71
5.1	Introduction	71
5.2	A recurrence	72

5.3	Use of a metric	74
5.4	Properties preserved by γ	75
5.5	Convergence	77
6	Summary of results	79
7	Simulation	80
7.1	Introduction	80
7.2	Growth rates	80
7.3	Increase of variances	81
7.4	The three-resistor networks	82
7.5	The length of the network	83
III	Spectra of Laplacians of random fractal graphs	85
8	Dirichlet-Neumann eigenvalues in the series-parallel graph	86
8.1	Introduction	86
8.2	A lower bound for the multiplicity of one particular eigenvalue	90
8.3	Near-localised eigenvalues	91
8.4	Simulation of eigenvalues for the series-parallel network	93
8.5	Cheeger constants	95

9 Eigenvalues of homogeneous random fractal graphs	101
9.1 Eigenvalues of the homogeneous series-parallel network	101
9.2 Spectral similarity	106
9.2.1 The multiplicities of the eigenvalues	108
9.2.2 More general results	110
10 Eigenvalues of general non-homogeneous graphs	112
10.1 Relationships between homogeneous and non-homogeneous spectra	112
10.2 Asymptotics for spectra of non-homogeneous fractal graphs . .	113
11 Some further examples	118
11.1 A random tree	118
11.2 Non-symmetric model graphs	119

Part I

Introduction

Chapter 1

Introduction and preliminaries

1.1 Random hierarchical systems

Given a space E , an index set R and a set of functions $f_r : E^k \rightarrow E; r \in R, k \geq 2$, with some probability measure μ on R , we can construct a random hierarchical system by starting with a sequence $\mathcal{X}^{(0)} = (X_j^{(0)}; j \in \mathbb{N})$ of independent identically distributed random variables, with distribution $\nu^{(0)}$ on E . We now take a family $Z_j^{(n)}; j \in \mathbb{N}, n \in \mathbb{N}$ of independent random variables on R distributed according to μ . We iteratively construct sequences $\mathcal{X}^{(n)} = (X_j^{(n)}; j \in \mathbb{N})$ for $n \in \mathbb{N}$ by setting

$$X_j^{(n)} = f_{Z_j^{(n)}}(X_{k(j-1)+1}^{(n-1)}, X_{k(j-1)+2}^{(n-1)}, \dots, X_{kj}^{(n-1)}).$$

This defines a sequence of probability distributions $(\nu^{(n)})_{n \in \mathbb{N}}$ on E , where $\nu^{(n)}$ is the law of $X_1^{(n)}$, obtained by iterating a map from the space of such distributions to itself. We will call this sequence a *random hierarchical system* on E with index set R , index distribution μ , functions $f_r; r \in R$ and initial distribution $\nu^{(0)}$.

We are interested in the asymptotics of this sequence: under what conditions are there non-degenerate limiting distributions, if so whether there is

a unique such (up to some scaling) and if the limit is a point mass whether the convergence of the random variables $X_1^{(n)}$ is almost sure.

A trivial example is given by $E = \mathbb{R}$, $R = \{1\}$ and $f_1(x_1, \dots, x_k) = \frac{x_1 + \dots + x_k}{k}$. In this case almost sure convergence of $X_1^{(n)}$ to a constant if $\mathbb{E}X_1^{(0)}$ is finite follows from the Strong Law of Large Numbers.

If we have $E = \mathbb{N}$, $R = [0, 1, \dots, k]$, and $f_i(x_1, \dots, x_k) = \sum_{j=1}^i x_j$, with $X_j^{(0)} = 1$ for all j , then the distribution of $X_1^{(n)}$ is the same as the distribution of the population size in the n th generation of a classical Galton-Watson branching process with offspring size bounded by k . In this case the limiting behaviour has been widely studied. If the branching process is supercritical (i.e. the offspring mean $\mu > 1$), then there exists a non-degenerate limiting distribution after normalisation (because the offspring distribution is bounded), while if the branching process is critical or subcritical (i.e. $\mu \leq 1$) then there is no non-degenerate limit and the branching process eventually dies out with probability 1. Much information on branching processes can be found in [4].

A generalisation of the branching process example is to consider functions which are weighted sums of k random variables. The iteration here is of the *smoothing transformation*. In [14] the case where $E = \mathbb{R}^+$ and R is some subset of $(\mathbb{R}^+)^k$, with, for $(w_1, \dots, w_k) \in R$,

$$f_{(w_1, \dots, w_k)}(x_1, \dots, x_k) = \sum_{i=1}^k w_i x_i,$$

is studied, and necessary and sufficient conditions for the existence of a non-degenerate fixed point of the iteration are obtained under moment conditions on the random weights. This is related to branching random walks. In [34], the natural extension to sums $\sum_{i=1}^{\infty} w_i x_i$ is considered, and again necessary and sufficient conditions for the existence of non-degenerate fixed points are obtained under moment conditions on the random weights.

The construction of random self-similar sets is an example of such an iteration where E is not a subset of \mathbb{R} . In this case E is a set of subsets of \mathbb{R}^n , and each function $f_r(X_1, \dots, X_k) = \cup_{j=1}^k \phi_r(X_j)$, where ϕ_i is a contraction on \mathbb{R}^n . The initial measure $\nu^{(0)}$ is usually taken to be a point mass on a particular subset.

Examples, including the random von Koch curve, are given in Chapter 15 of [17], and the proofs of the existence of limiting distributions are given in [16, 19, 37]. In [37] the Hausdorff dimension for a class of such random fractals is obtained. The construction of random fractal measures is a further example, where E is now a set of measures. Given a contraction condition, the existence of a limiting random measure for classes of iterations is shown in [26, 27]. The contraction mapping theorem is an important tool in the proofs of these results.

Similar iterations are used when considering homogenisation on nested fractals in [24, 31]. Here E is a cone of Dirichlet forms, $R = \{1\}$ (for deterministic fractals), and f_1 is a renormalisation map. Here the initial distribution $\nu^{(0)}$ can be any distribution on E . The problem of finding a random Laplacian on a random fractal is a similar problem, but with more general R and the initial distribution a point mass.

Further examples arise naturally in the study of the renormalisation of networks of random resistors on hierarchical lattices, see for example [15, 44, 45, 51, 52]. Hierarchical models are studied in statistical physics because they can often be solved explicitly and may provide insight into behaviour on standard lattices. For a discussion see [7, 48] and for studies of some statistical physics models in the hierarchical setting see [13, 20]. Similar models also occur in other situations arising in physics, such as modelling earthquakes and fibre strengths in [40]. A discussion of the relationship between classical probability results such as the central limit theorem and renormalisation in physics appears in [28]. Fluctuation theorems for specific models are obtained in [15, 44, 52].

An application to computer science is the study of the biased coin problem considered in [3, 9] and related to the use of physical sources of randomness with a small bias. A model of the above type is studied, with $R = \{1\}$ and

$$f_1(x_1, x_2) = \min \left(\left(\frac{1}{2} - \epsilon \right) x_1 + \left(\frac{1}{2} + \epsilon \right) x_2, \left(\frac{1}{2} + \epsilon \right) x_1 + \left(\frac{1}{2} - \epsilon \right) x_2 \right).$$

Another example from the computer science literature, discussed in [2, 9, 10] is concerned with the problem of a group of processors designating a processor as a leader in circumstances where some of the processors are faulty. The

processors are arranged as nodes of a binary tree and the aim is to elect a non-faulty processor. Here $R = \{1, 2\}$ and the functions are

$$\begin{aligned} f_1(x_1, x_2) &= \frac{x_1 + x_2}{2} \\ f_2(x_1, x_2) &= \min(x_1, x_2) \end{aligned}$$

with probabilities $1 - \epsilon$ and ϵ respectively, where ϵ is the probability of an individual processor being faulty, with processors being faulty independently of each other and of the initial sequence of random variables $\mathcal{X}^{(0)}$. The random variables $X_j^{(n)}$ represent the probability of electing a non-faulty leader. Using an extended moment calculation, it was shown in [10] that, for $\epsilon < \frac{1}{2}$, there exists δ_1 such that $\mathbb{P}(X_j^{(n)} < \delta_1) < \delta_2$ for n large enough, for any $\delta_2 > 0$.

In the case where $R = \{1\}$ and f_1 is continuous, defined on $\mathcal{D} = [0, 1]$, concave, positively homogeneous and increasing, convergence in probability to a constant is proved in [47]. For one particular function obtained by considering a specific lattice, almost sure convergence to a constant is proved in [15]. Further work is in [32]. In Chapter 2 we build on this work to obtain almost sure convergence to a constant for a subclass of iterations where E is a bounded below and closed domain in \mathbb{R} and the functions f_r are all bounded above by a weighted mean. Chapter 2 is based on [29].

In Chapter 3 we will consider random fractal resistor networks. These are essentially a randomised version of the hierarchical lattices. We will consider a specific example, which we call the *series-parallel* network, in some depth, and consider the asymptotics of the resistance. We will also consider a different property of this network, the distance between the two boundary points. In Chapter 4 we will show that in some cases there is a family of fixed points of a random hierarchical system describing this, and that the distance converges in distribution to one of them.

In Section 3.3 we consider some further resistor networks, and discuss their asymptotics. In Chapter 5 we will discuss another hierarchical system with a non-degenerate fixed point. Our results on random hierarchical systems are summarised in Chapter 6 and where there are open questions, the results of some computer simulations are shown in Chapter 7.

1.2 Random fractal graphs

The hierarchical lattices already mentioned are examples of fractal graphs, and the random resistor networks are examples of random fractal graphs. We investigate further properties of such graphs, especially the spectral properties of their Laplacians.

The graphs that we will consider are related to self-similar fractals. Random self-similar fractals are constructed in [16, 19, 37], and random self-similar fractal measures are constructed in [26, 27]. Random fractals are also discussed in Chapter 15 of [17]. A Brownian motion on a random Sierpiński gasket is constructed using resistance-based methods in [22], and the asymptotic behaviour of its eigenvalue counting function is discussed in [23].

Homogeneous fractals related to the homogeneous graphs discussed below are studied in [5, 21].

Self-similar fractal graphs are studied as approximations to geometric fractals, and are used in the construction of Brownian motion on deterministic fractals such as the Sierpiński gasket, described in Chapters 2 and 3 of [30]. The relationship between large-scale structures such as fractal graphs and the small scale of fractals is discussed in [49].

We will consider four different types of construction of random fractal graphs. *Homogeneous* graphs have much of the symmetry associated with deterministic fractals, while *non-homogeneous* graphs do not, but have averaging properties which may enable almost-sure results to hold. Associated with each of these types is a construction by *replacement*, and also a construction by *duplication*. The replacement constructions are generalisations to the random case of the construction described in Definition 5.2 of [36].

To allow a slightly more general class of graphs, we will allow our graphs to have multiple edges. We fix $M \in \mathbb{N}$, and consider a *cell* to be a copy of the complete graph on M vertices.

Given a graph G , let $V(G)$ be the vertex set of G , and let $E(G)$ be the edge set. We will assume the following.

Assumption 1.1. *We have a connected graph G satisfying:*

- *There exist $c(G)$ sets $C_1(G), \dots, C_{c(G)}(G) \subseteq V(G)$ of M vertices each.*
- *Each $v \in V(G) \in C_j(G)$ for at least one j .*
- *Given $u, v \in V(G)$, the number of edges in $E(G)$ connecting u and v is $w(u, v) = |\{j : u, v \in C_j(G)\}|$, the number of cells containing both u and v .*
- *There is a set $\partial G \in V(G)$ of M vertices, which we will call the boundary of G .*

For an example see Figure 1.1.

Definition 1.2. *Given two graphs G_0 and G_1 satisfying Assumption 1.1 (with the same M), to replace cell j of G_0 with G_1 , we define a bijection $\eta : C_j(G_0) \rightarrow \partial G_1$, and obtain the vertex set of a new graph G_2 by taking $V(G_0) \cup V(G_1)$ and identifying each $v \in C_j(G_0)$ with $\eta(v)$. The boundary ∂G_2 is the same as that of G_0 .*

We set $c(G_2) = c(G_0) + c(G_1) - 1$, and define cells

- $C_i(G_2) = C_i(G_0)$, for $i \in \{1, 2, \dots, c(G_0)\} \setminus \{j\}$
- $C_j(G_2) = C_{c(G_1)}(G_1)$
- $C_{c(G_0)+i}(G_2) = C_i(G_1)$ for $i \in \{1, 2, \dots, c(G_1) - 1\}$.

Then, if $w_i(u, v)$ is the number of edges connecting u and v in G_i , we set

$$w_2(u, v) = w_1(u, v) + w_0(u, v) - I_{\{u, v \in C_j(G_0)\}}$$

so that the graph G_2 satisfies Assumption 1.1.

If the replacing graph G_1 is not symmetric with respect to its boundary, then the structure of the new graph depends on the choice of the bijection.

The assumption that we only identify boundary vertices is a finite ramification assumption, and excludes graphs related to non-finitely ramified fractals such as the Sierpiński carpet.

We will consider a fixed set of graphs $\{\Gamma_r; r \in R\}$, satisfying Assumption 1.1, for some finite index set R .

Definition 1.3. *To form the homogeneous replacement sequence, we start with a graph F_0 consisting of a single cell. We have a sequence $\{Z_n; n \in \mathbb{N}\}$ with values in R . We proceed inductively, constructing F_n from F_{n-1} by replacing each cell of the latter with a copy of Γ_{Z_n} .*

To form the homogeneous duplication sequence we again start with a single cell, F'_0 , and we again have a sequence $\{Z_n; n \in \mathbb{N}\}$ with values in R . We define F'_n by taking the graph Γ_{Z_n} and replacing each of its cells by identical copies of F'_{n-1} .

For the non-homogeneous sequences, we define a family of random variables $\{Z_j^{(n)}; n, j \in \mathbb{N}\}$ on R . For the replacement sequence, $G^{(0)}$ is a single cell. To construct $G^{(n)}$ from $G^{(n-1)}$, we replace the cell $C_j(G^{(n-1)})$ with a copy of $\Gamma_{Z_j^{(n)}}$.

For the duplication sequence, we define a family of graphs $\{G_j^{(n)}; n, j \in \mathbb{N}\}$ and we start with $G_j^{(0)}$ being a single cell for all $j \in \mathbb{N}$. To form $G_j^{(n)}$ we take the graph $\Gamma_{Z_j^{(n)}}$, let σ be a random permutation of $[1, \dots, c(\Gamma_{Z_j^{(n)}})]$, and replace the cell $C_{\sigma(i)}(\Gamma_{Z_j^{(n)}})$ with $G_{(j-1)k+i}^{(n-1)}$, where $k = \max_{r \in R} c(\Gamma_r)$.

We will take the sequence $\{Z_n; n \in \mathbb{N}\}$ and the family $\{Z_j^{(n)}; n, j \in \mathbb{N}\}$ to consist of independent identically distributed random variables.

The non-homogeneous duplication sequence can be considered as a random hierarchical system, with E a space of graphs and the functions f_r corresponding to replacing the cells of the model graph Γ_r with the input graphs.

The following two lemmas show that the graphs produced by the replacement and duplication constructions generally have the same distributions.

Lemma 1.4. *If $\{Z_n; n \in \mathbb{N}\}$ are independent identically distributed random variables, then the distributions of the graphs F_n and F'_n are the same for all n .*

Proof. This is true by definition for $n = 0$. Proceeding inductively, F_n is formed by replacing each cell of F_{n-1} with a copy of Γ_{Z_n} . By the stationarity property, $(Z_{n-m+1}, Z_{n-m+2}, \dots, Z_n)$ has the same distribution as (Z_1, Z_2, \dots, Z_m) , and hence F_n can be formed from F_{n-m} by replacing each cell with a copy of a graph whose distribution is the same as that of F_m . Letting $m = n - 1$ tells us that F_n can be formed from F_1 by replacing each cell by a graph of the same distribution as F_{n-1} . But F_1 has the same distribution as Γ_{Z_n} , and hence if F_{n-1} has the same distribution as F'_{n-1} the same follows for F_n and F'_n . \square

Lemma 1.5. *If $\{Z_j^{(n)}; n, j \in \mathbb{N}\}$ are independent identically distributed random variables, then the distributions of the graphs $G^{(n)}$ and $G_j^{(n)}$ are the same, for all n and j .*

Proof. The proof is essentially the same as for the homogeneous case. That is, $G^{(n)}$ can be obtained by replacing the cells of $G^{(n-m)}$ by independent copies of $G^{(m)}$. We take $m = n - 1$ and use the fact that $G^{(1)}$ has the same distribution as $\Gamma_{Z_j^{(n)}}$. \square

Examples of the deterministic version include nested fractal graphs such as the Sierpiński pre-gasket. An early stage of the construction of the Sierpiński gasket is shown in Figure 1.2. In this case $M = 3$, $R = \{1\}$, and the model graph Γ_1 is shown in Figure 1.1.

A randomised version of this corresponds to the random gasket described in [22]. Here $R = \{1, 2\}$, and the model graphs are the two shown in Figure 1.1. The graph Γ_1 has 3 cells, and Γ_2 has 6 cells.

An early stage of the construction of the corresponding random fractal is shown in Figure 1.3.

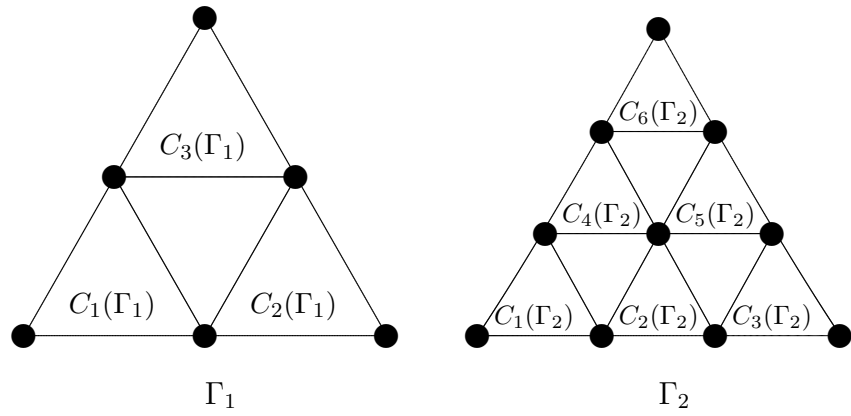


Figure 1.1: The gasket model graphs, with the cells $C_i(\Gamma_j)$ labelled

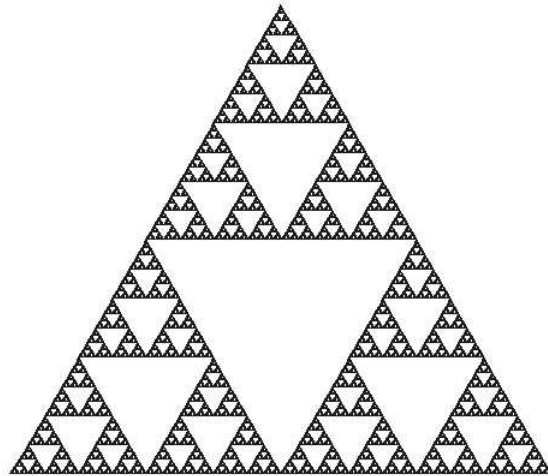


Figure 1.2: The Sierpiński gasket

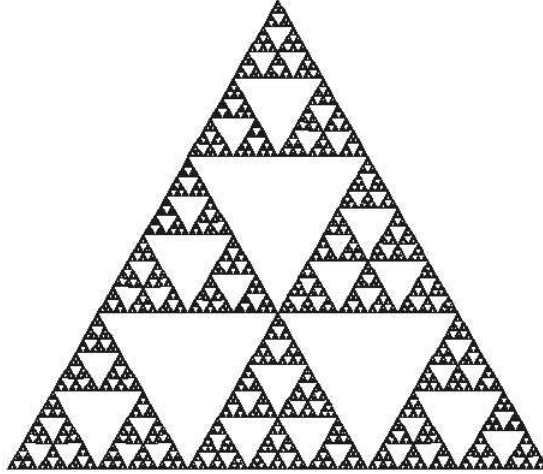


Figure 1.3: A randomised Sierpiński gasket

We will study the eigenvalues of the graph Laplacians of these graphs, with the aim of comparing the behaviour of the spectra with the behaviour of the spectra of deterministic fractals and fractal graphs.

For the definitions of the Laplacian and eigenvalues we follow [11].

Definition 1.6. *Given a graph G (which may have multiple edges), we set $V(G)$ to be the set of vertices of G . Now for $u, v \in V(G)$ define $\delta(v)$ to be the degree of v in G , and $w(u, v)$ to be the number of edges connecting u and v . We now define the Laplacian \mathcal{L} by*

$$\mathcal{L}(u, v) = \begin{cases} 1 & u = v \\ \frac{-w(u, v)}{\sqrt{\delta(u)\delta(v)}} & u, v \text{ adjacent} \\ 0 & \text{otherwise} \end{cases}$$

and we will define the eigenvalues of G to be those of \mathcal{L} .

These are the eigenvalues studied in [11], from which we see that if λ is an eigenvalue then $0 \leq \lambda \leq 2$. There is always an eigenvalue 0, but 2 is an eigenvalue if and only if the graph has a non-trivial bipartite component. Note that our graphs are always connected. For most of the non-homogeneous

examples we discuss, they are not in general bipartite, although $G^{(n)}$ may be bipartite for small n . However homogeneous random fractal graphs will be bipartite if the model graphs Γ_i are bipartite for all i .

Note that the Laplacian is sometimes defined slightly differently - for example Definition 4.5 of [36]. However the spectrum differs by at most a simple transformation. For example λ is an eigenvalue of the Laplacian as defined above if and only if $-\lambda$ is an eigenvalue of the Laplacian as defined in [36], with the same multiplicity.

Also, different definitions of the eigenvalues of graphs are sometimes found, such as those defined in [12] and those calculated in [18].

For a graph G , we can now define a *spectral measure* by

$$\mu_G(A) = \frac{|\{\lambda \in A : \lambda \text{ eigenvalue of } \mathcal{L}\}|}{|V(G)|}$$

(counting multiple eigenvalues) for subsets $A \subseteq [0, 2]$. Given a sequence of graphs G_n we can study the sequence of spectral measures. In some cases there will be a limiting spectrum, but if the sequence of graphs is random the convergence may be in distribution.

We are particularly interested in the set of points $\Lambda \subseteq [0, 2]$ such that, for $\lambda \in \Lambda$, $\limsup_{n \rightarrow \infty} \mu_{G_n}\{\lambda\} > 0$ and in the question of when $\lim_{n \rightarrow \infty} \mu_{G_n}(\Lambda) = 1$, so that for large n the spectrum is dominated by such eigenvalues. If this applies then we will describe the limiting behaviour as *pure point*. In this case any subsequential limit of $(\mu_{G_n})_{n \in \mathbb{N}}$ will be a pure point measure as defined, for example, in [42].

In the case of deterministic fractals, localised eigenfunctions were found for the Sierpiński gasket in the physics literature [41]. In the context of fractals *localised* means that the eigenfunctions are strictly localised - they are supported on a compact subset of the fractal. Elsewhere in the physics literature it usually means that the eigenfunction decays exponentially away from a particular region.

The Sierpiński gasket also has a property known as *spectral decimation* [41]. This allows many eigenvalues of $G^{(n)}$ to be obtained as roots of $R(z) = \lambda$,

where λ is an eigenvalue of $G^{(n-1)}$ and $R(z)$ is a quadratic function. As a result the limiting spectrum is closely related to the Julia set of $R(z)$. Further details are in Section 9.2. A rigorous mathematical treatment of this is in [46]. In [50], it is shown that the Neumann spectrum of the Sierpiński gasket is pure point. Spectral decimation needs strong conditions to work for fractals other than the Sierpiński gasket - these are described in [36].

For fractal graphs, the existence of a pure point localised spectrum for many two-point graphs (i.e. those where there are just two boundary points - in our above notation $M = 2$) was shown in [35]. In [36] the method of spectral decimation is developed to apply to a large class of highly symmetric graphs, and the specific application to the graphs related to the Sierpiński gasket (known as the Sierpiński pre-gaskets) is described in detail in [50].

For nested fractals, a class of self-similar fractals with a strong symmetry condition introduced in [33], the existence of a pure point spectrum generated by localised eigenfunctions is shown in [43]. However, the conditions needed for spectral decimation to give details of the eigenvalues are quite strong. Examples where spectral decimation does not work include the pentagasket, whose spectral properties are studied in detail in [1].

Some examples of graphs with spectra similar in some respects to those found for fractal graphs, consisting of isolated points together with a set derived from the Julia set of a polynomial, have been found for some graphs associated with fractal groups. This is discussed in [6].

The graphs are also examples of random graphs, and we can compare the spectral properties we find with those for other types of random graphs. These are considered in [18], where the eigenvalues considered are those of the adjacency matrix rather than the Laplacian of [11]. For Erdős-Rényi random graphs, there is a semi-circle law. In [18] this is contrasted with the small-world model, for which simulations suggest that there can be sharp local maxima in the spectrum, but there are no actual singularities similar to those produced by localised eigenfunctions. Another example, also discussed in [18], is the scale-free graph. In this case simulation of the spectrum of the adjacency matrix shows a triangular form with a high number of eigenvalues with small modulus. In [18] localisation of the eigenvectors is suggested as a possible explanation for this.

More on random graphs can be found in [8].

In Chapter 8, we concentrate on one example, the non-homogeneous series-parallel graph, and show that its Laplacian has localised (specifically Dirichlet-Neumann, as used in [43]) eigenvalues, and we obtain lower bounds for the proportion of the spectrum that they form. We also briefly discuss eigenfunctions which are almost localised, in the sense that they are very close to zero on a large part of the graph. This appears to be due to specific properties of this graph. Finally we consider the Cheeger constants, which are related to the eigenvalues and also to the connectivity of the graph.

In Chapter 9, we discuss how the methods of [36] can be applied to homogeneous fractal graphs, and describe what happens to the homogeneous series-parallel graph.

In Chapter 10 we prove some general results about the asymptotics of the spectra of non-homogeneous self-similar graphs. Finally in Chapter 11 we discuss some specific examples and show results of some simulations of the spectra. We also discuss an interesting deterministic fractal graph which does not satisfy the conditions of [36].

Part II

Asymptotics for random hierarchical systems

Chapter 2

Almost-sure convergence

2.1 Introduction

In this chapter, we describe results on random hierarchical systems where the functions f_i are defined on a domain $\mathcal{D} \subseteq \mathbb{R}$, and prove that a subclass of these iterations converges almost surely to a constant.

In the case where the function is continuous, defined on $\mathcal{D} = [0, 1]$, concave, positively homogeneous and increasing, convergence in probability to a constant is proved in [47]. For one particular function obtained by considering a specific hierarchical lattice, almost sure convergence to a constant is proved in [15]. The most general result to date is in [32] who at each level n of the renormalisation considered a (deterministic) function f_n of k_n variables. They assumed that the initial sequence of random variables was stationary and m -dependent, i.e. the two collections $\{X_1^{(0)}, \dots, X_n^{(0)}\}$ and $\{X_{n+m+1}^{(0)}, X_{n+m+2}^{(0)}, \dots\}$ are independent, and proved a weak law of large numbers. Under stronger conditions they proved the following strong law (Theorem 2.1(iii) of [32]).

Theorem. *(Li and Rogers) Suppose that the f_n satisfy the subadditive con-*

straint

$$f_n(x_1, \dots, x_k) \leq \frac{1}{k} \sum_{i=1}^k x_i \text{ for all } (x_1, \dots, x_k) \in \mathcal{D}^k.$$

Then if \mathcal{D} is bounded below, the f_n , $n \geq 1$ are symmetric functions of $k(\geq 2)$ real variables and $\mathcal{X}^{(0)} = \{X_j^{(0)}; j \geq 1\}$ is a sequence of i.i.d. random variables such that

$$\mathbb{E}|X_1^{(0)}| < \infty,$$

then for some $\lambda \in \mathcal{D}$,

$$\lim_{n \rightarrow \infty} X_1^{(n)} = \lambda \text{ a.s.}$$

We will develop a randomised version of the problem and extend the weak law from [32] to this setting. In the randomised setting the symmetry condition needed for the strong law is less rigid and we can then use this strong law to weaken the conditions of [32] in the non-randomised model, giving the following result.

Theorem 2.1. (a) If

$$f(x_1, \dots, x_k) \leq \sum_{i=1}^k \alpha_i x_i$$

where the α_i are positive constants with $\alpha_i < 1$ for each i , $\sum_{i=1}^k \alpha_i = 1$ and

$$\mathbb{E} \left(|X_1^{(0)}| (L(|X_1^{(0)}|))^{\delta} \right) < \infty$$

where

$$\delta > 1 \text{ and } L(x) = \max\{1, \log x\},$$

then there exists $\lambda \in \mathcal{D}$ with

$$X_1^{(n)} \rightarrow \lambda \text{ a.s.}$$

(b) Further, if $\alpha_i = \frac{1}{k}$ for all i , then the conclusion holds under the weaker condition that $\mathbb{E}|X_1^{(0)}| < \infty$.

Part (b) of this was originally claimed in [51].

The main results and their proofs are described in Section 2.2. Some lemmas necessary to prove the results are included in Section 2.3.

For recent work on this problem in the Banach space setting, see [25].

2.2 The randomised model

We let $\mathcal{X}^{(0)} = \{X_j^{(0)}; j \geq 1\}$ be a sequence of independent identically distributed random variables with $\mathbb{P}(X^{(0)} \in \mathcal{D}) = 1$ for some closed domain $\mathcal{D} \subseteq \mathbb{R}$. We define $\xi = \inf\{x : x \in \mathcal{D}\}$. (Note that ξ may be $-\infty$ but otherwise $\xi \in \mathcal{D}$.)

Now let $Z_j^{(n)}; j \geq 1, n \geq 1$ be a set of independent (of each other and of $\mathcal{X}^{(0)}$) and identically distributed random variables taking values in some measurable space R . For each $r \in R$ let $f_r : \mathcal{D}^k \rightarrow \mathcal{D}$ be a real measurable function of k variables, where k is a fixed integer, $k > 1$.

We now define a sequence $\{\mathcal{X}^{(n)}; n \geq 0\}$ by

$$X_j^{(n)} = f_{Z_j^{(n)}}(X_{(k-1)j+1}^{(n-1)}, X_{(k-1)j+2}^{(n-1)}, \dots, X_{kj}^{(n-1)}),$$

$$\mathcal{X}^{(n)} = \mathcal{R}\mathcal{X}^{(n-1)} = \{X_j^{(n)}; j \geq 1\}.$$

The leader election model of [2, 9, 10] has $R = \{1, 2\}$, $f_1(x_1, x_2) = \frac{x_1+x_2}{2}$, $f_2(x_1, x_2) = \min\{x_1, x_2\}$.

Define σ -algebras $\mathcal{F}_n = \sigma(X_i^{(j)}; 0 \leq j \leq n, i \geq 1)$ and $\mathcal{G}_n = \sigma(Z_i^{(n)}; i \geq 1)$.

The following result is based on Theorem 2.1(i) of [32] and the proof is largely the same except for the use of conditional expectation.

Theorem 2.2. *Suppose that we have non-negative constants $\alpha_{i,r}, 1 \leq i \leq$*

$k, r \in R$, such that

$$\sum_{i=1}^k \alpha_{i,r} = 1 \text{ for all } r \text{ and } \alpha_{i,r} \leq A \text{ for all } i, r \text{ and some constant } A < 1.$$

Further suppose that for all $r \in R$, we have the subadditivity constraint

$$f_r(x_1, x_2, \dots, x_k) \leq \sum_{i=1}^k \alpha_{i,r} x_i.$$

If

$$\mathbb{E}(X_1^{(0)} \vee 0) < \infty,$$

then for some $\lambda \in \mathcal{D} \cup \{\xi\}$,

$$\mathbb{E}X_1^{(n)} \downarrow \lambda \text{ as } n \rightarrow \infty \text{ and } X_1^{(n)} \rightarrow_p \lambda.$$

Proof. Define

$$g_j^{(n)}(x_1, \dots, x_k) = \sum_{i=1}^k \alpha_{i, Z_j^{(n)}} x_i$$

and

$$\begin{aligned} h_j^{(n)}(x_1, \dots, x_{k^n}) &= g_j^{(n)}(h_{(j-1)k+1}^{(n-1)}(x_1, \dots, x_{k^{n-1}}), \\ &h_{(j-1)k+2}^{(n-1)}(x_{k^{n-1}+1}, \dots, x_{2k^{n-1}}), \dots, \\ &h_{jk}^{(n-1)}(x_{(k-1)k^{n-1}+1}, \dots, x_{k^n})) \end{aligned}$$

where $h_j^{(0)}(x) = x$ for all j .

Note that

$$h_j^{(n)}(x_1, \dots, x_{k^n}) = \sum_{i=1}^{k^n} d_{i,j}^{(n)} x_i$$

where $d_{i,j}^{(n)} = \alpha_{p, Z_j^{(n)}} d_{q, (j-1)k+q}^{(n-1)}$ where $i = (p-1)k^{n-1} + q$ with $1 \leq p \leq k$ and $1 \leq q \leq k^{n-1}$. Inductively, we see that $\max_i d_{i,1}^{(n)} \leq A^n \rightarrow 0$ as $n \rightarrow \infty$. This allows us to use Lemma 2.7 (with $a_{n,i} = d_{i,1}^{(n)}$) to give

$$\sum_{i=1}^{k^n} d_{i,1}^{(n)} X_i^{(0)} \rightarrow_p \mathbb{E}X_1^{(0)}. \quad (2.1)$$

(with convergence to $-\infty$ if $\mathbb{E}X_1^{(0)} = -\infty$)

Now

$$\begin{aligned}\mathbb{E}(X_1^{(n)}|\mathcal{F}_{n-1}, \mathcal{G}_n) &= f_{Z_1^{(n)}}(X_1^{(n-1)}, \dots, X_k^{(n-1)}) \\ &\leq \sum_{i=1}^k \alpha_{i, Z_1^{(n)}} X_i^{(n-1)}\end{aligned}$$

and using properties of conditional expectation and the i.i.d. property of $\mathcal{X}^{(n-1)}$ we have

$$\begin{aligned}\mathbb{E}(X_1^{(n)}|\mathcal{G}_n) &\leq \sum_{i=1}^k \alpha_{i, Z_1^{(n)}} \mathbb{E}(X_i^{(n-1)}|\mathcal{G}_n) \\ &= \mathbb{E}(X_1^{(n-1)}|\mathcal{G}_n)\end{aligned}$$

but $X_1^{(n-1)}$ is independent of \mathcal{G}_n so that $\mathbb{E}X_1^{(n)}, n \geq 1$ is a non-increasing sequence of real numbers, hence there exists $\lambda \in [-\infty, \infty)$ such that

$$\mathbb{E}X_1^{(n)} \downarrow \lambda \text{ as } n \rightarrow \infty. \quad (2.2)$$

Then if $\mathbb{E}X_1^{(0)} = -\infty$ we have, using (2.1), that $X_1^{(n)} \rightarrow_p -\infty$ and if $\mathbb{E}|X_1^{(0)}| < \infty$ then

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_1^{(n)} \geq \mathbb{E}X_1^{(0)} + \epsilon) = 0.$$

Further,

$$X_1^{(n)} I_{\{X_1^{(n)} \geq \mathbb{E}X_1^{(0)} + \epsilon\}} \leq \sum_{i=1}^{k^n} d_{i,1}^{(n)} |X_i^{(0)}| I_{\{\sum_{i=1}^{k^n} d_{i,1}^{(n)} X_i^{(0)} \geq \mathbb{E}X_1^{(0)} + \epsilon\}},$$

and for every $M > 0$,

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^{k^n} d_{i,1}^{(n)} |X_i^{(0)}| I_{\{\sum_{i=1}^{k^n} d_{i,1}^{(n)} X_i^{(0)} \geq \mathbb{E}X_1^{(0)} + \epsilon\}}\right) \\ \leq M \mathbb{P}\left(\sum_{i=1}^{k^n} d_{i,1}^{(n)} X_i^{(0)} \geq \mathbb{E}X_1^{(0)} + \epsilon\right) + \mathbb{E}\left(\sum_{i=1}^{k^n} d_{i,1}^{(n)} |X_i^{(0)}| I_{\{\sum_{i=1}^{k^n} d_{i,1}^{(n)} |X_i^{(0)}| \geq M\}}\right)\end{aligned}$$

which with $\mathbb{E}|X_1^{(0)}| < \infty$ implies that

$$\lim_{n \rightarrow \infty} \mathbb{E}(X_1^{(n)} I_{\{X_1^{(n)} \geq \mathbb{E}X_1^{(0)} + \epsilon\}}) = 0. \quad (2.3)$$

Now we repeat these arguments but starting with $\mathcal{X}^{(n_0)}$ instead of with $\mathcal{X}^{(0)}$. The conditions on $\mathcal{X}^{(0)}$ imply that $\mathbb{E}(X_1^{(n_0)} \vee 0) < \infty$ so they apply to $\mathcal{X}^{(n_0)}$ as well. So if for any n_0 , $\mathbb{E}X_1^{(n_0)} = -\infty$ then $X_1^{(n)} \rightarrow_p -\infty$ and if $\mathbb{E}|X_1^{(n_0)}| < \infty$ then

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_1^{(n)} \geq \mathbb{E}X_1^{(n_0)} + \epsilon) = 0$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E}(X_1^{(n)} I_{\{X_1^{(n)} \geq \mathbb{E}X_1^{(n_0)} + \epsilon\}}) = 0. \quad (2.4)$$

So if $\lambda = -\infty$ then we have

$$X_1^{(n)} \rightarrow_p -\infty$$

and if $\lambda > -\infty$, for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_1^{(n)} \geq \lambda + \epsilon) = 0$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E}(X_1^{(n)} I_{\{X_1^{(n)} \geq \lambda + \epsilon\}}) = 0. \quad (2.5)$$

If $\lambda > -\infty$ then

$$\begin{aligned} \mathbb{E}(X_1^{(n)} - \lambda) &= \mathbb{E}\left((X_1^{(n)} - \lambda) I_{\{X_1^{(n)} \geq \lambda + \epsilon'\}}\right) \\ &\quad + \mathbb{E}\left((X_1^{(n)} - \lambda) I_{\{X_1^{(n)} \leq \lambda - \epsilon'\}}\right) + \mathbb{E}\left((X_1^{(n)} - \lambda) I_{\{\lambda - \epsilon' < X_1^{(n)} < \lambda + \epsilon'\}}\right), \end{aligned}$$

so using (2.2) and (2.5)

$$\liminf_{n \rightarrow \infty} \mathbb{E}\left((X_1^{(n)} - \lambda) I_{\{X_1^{(n)} \leq \lambda - \epsilon'\}}\right) \geq -\limsup_{n \rightarrow \infty} \mathbb{E}\left((X_1^{(n)} - \lambda) I_{\{\lambda - \epsilon' < X_1^{(n)} < \lambda + \epsilon'\}}\right) \geq -\epsilon'$$

for all $\epsilon' > 0$, while obviously

$$\limsup_{n \rightarrow \infty} \mathbb{E}\left((X_1^{(n)} - \lambda) I_{\{X_1^{(n)} \leq \lambda - \epsilon'\}}\right) \leq 0.$$

Hence

$$\lim_{n \rightarrow \infty} \mathbb{E}\left((X_1^{(n)} - \lambda) I_{\{X_1^{(n)} \leq \lambda - \epsilon'\}}\right) = 0 \quad (2.6)$$

and (2.5) and (2.6) give the result

$$X_1^{(n)} \rightarrow_p \lambda \text{ as } n \rightarrow \infty.$$

As in [32], the proof finishes by taking an a.s.-convergent subsequence to obtain $\lambda \in \mathcal{D} \cup \{\xi\}$. \square

The next result is based on Theorem 2.1(ii) of [32] and gives a form of strong law, but only for the lim sup. Again the proof is based on that in [32].

Lemma 2.3. *With the hypotheses of Theorem 2.2 and additionally if \mathcal{D} is bounded below and we have*

$$\mathbb{E}(|X_1^{(0)}|(L(|X_1^{(0)}|))^\delta) < \infty$$

for some $\delta > 1$ then for some $\lambda \in \mathcal{D}$ the conclusion of Theorem 2.2 holds and additionally

$$\limsup_{n \rightarrow \infty} X_1^{(n)} = \lambda \text{ almost surely.}$$

Proof. With the hypotheses of Theorem 2.2 holding, we know that $X_1^{(n)} \rightarrow_p \lambda$ for some $\lambda \in \mathcal{D} \cup \{\xi\}$. However \mathcal{D} is closed and bounded below, so $\xi \in \mathcal{D}$, hence $\lambda \in \mathcal{D}$.

There exists a subsequence $\{n_j; j \geq 1\}$ such that

$$\lim_{j \rightarrow \infty} X_1^{(n_j)} = \lambda \text{ a.s.}$$

and hence

$$\limsup_{j \rightarrow \infty} X_1^{(n_j)} \geq \lambda \text{ a.s.} \tag{2.7}$$

Now $X_1^{(n)} \leq \sum_{i=1}^{k^n} d_{i,1}^{(n)} X_i^{(0)}$ as before. The condition $\sum_{i=1}^{k^n} (d_{i,1}^{(n)})^2 \leq cb^{-n}$ ($b > 1, c > 0$) necessary to apply Lemma 2.8 is easy in our case (see Example 2.1 of [32]). So we have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{k^n} d_{i,1}^{(n)} X_i^{(0)} = \mathbb{E}X_1^{(0)} \text{ a.s.}$$

and hence

$$\limsup_{n \rightarrow \infty} X_1^{(n)} \leq \mathbb{E}X_1^{(0)} \text{ a.s.} \quad (2.8)$$

As in the proof of Theorem 2.2, we now observe that the conditions on $\mathcal{X}^{(0)}$ imply the same conditions on $\mathcal{X}^{(n_0)}$ for any integer n_0 (we need \mathcal{D} to be bounded below here, and also Jensen's inequality). So we can use identical arguments to those above to obtain

$$\limsup_{n \rightarrow \infty} X_1^{(n)} \leq \mathbb{E}X_1^{(n_0)} \text{ a.s.} \quad (2.9)$$

and letting $n_0 \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} X_1^{(n)} \leq \lambda \text{ a.s.} \quad (2.10)$$

Then (2.7) and (2.10) give the result. \square

The next result is based on Theorem 2.1(iii) of [32] although the symmetry condition is slightly modified. It gives strong-law convergence in a restricted case.

Lemma 2.4. *Assume the hypotheses of Lemma 2.3, and further the following symmetry condition:*

For all $\sigma \in S_k$, the permutation group on k elements, we have

$$\mathbb{P}(f_{Z_i^{(n)}} \in E) = \mathbb{P}(f_{Z_i^{(n)}} \circ \sigma \in E)$$

for all \mathbb{P} -measurable subsets E of measurable functions of k real variables (i.e. changing the order of the input variables does not alter the conditional distributions of $\mathcal{X}^{(n-1)}$ and $\mathcal{X}^{(n)}$).

Then for some $\lambda \in \mathcal{D}$, we have

$$\lim_{n \rightarrow \infty} X_1^{(n)} = \lambda \text{ a.s.}$$

Further, if $\alpha_{i,r} = 1/k$ for all i, r , then the conclusion holds under the weaker condition that $\mathbb{E}|X_1^{(0)}| < \infty$.

Proof. The hypotheses of Lemma 2.3 apply so we know that there exists a $\lambda \in \mathcal{D}$ such that

$$X_1^{(n)} \rightarrow_p \lambda$$

and

$$\limsup X_1^{(n)} = \lambda \text{ a.s.} \quad (2.11)$$

Similarly for every $l > 1$ we have $X_l^{(n)} \rightarrow_p \lambda$ and $\limsup X_l^{(n)} = \lambda$ a.s.

Now set

$$Y^{(n)} = \sum_{i=1}^k X_i^{(n)}.$$

For $n \geq 0$, define σ -algebras

$$\mathcal{F}'_n = \sigma(Y^{(l)}; l \geq n).$$

Our symmetry condition gives that, for $1 \leq i \leq k$,

$$\mathbb{E}(X_{(i-1)k+1}^{(n)} + \cdots + X_{ik}^{(n)} | \mathcal{F}'_{n+1}) = \mathbb{E}(Y^{(n)} | \mathcal{F}'_{n+1})$$

and hence

$$\mathbb{E}(Y^{(n)} | \mathcal{F}'_{n+1}) = \frac{\mathbb{E}(X_1^{(n)} + \cdots + X_{k^2}^{(n)} | \mathcal{F}'_{n+1})}{k}.$$

We note that the symmetry condition implies that $\mathbb{E}\alpha_{i,Z_j^{(n+1)}} = 1/k$ and further that $Z_j^{(n+1)}$ is independent of \mathcal{F}'_{n+1} , and so

$$\mathbb{E}(\alpha_{i,Z_j^{(n+1)}} | \mathcal{F}'_{n+1}) = \frac{1}{k}.$$

Hence conditioning on the $Z_j^{(n+1)}$ and using the subadditivity of the functions will give

$$\mathbb{E}\left(\frac{X_1^{(n)} + \cdots + X_k^{(n)}}{k} \middle| \mathcal{F}'_{n+1}\right) \geq \mathbb{E}(X_1^{(n+1)} | \mathcal{F}'_{n+1})$$

and similarly for the other $X_i^{(n+1)}$, giving

$$\begin{aligned} \mathbb{E}(Y^{(n)} | \mathcal{F}'_{n+1}) &\geq \mathbb{E}(X_1^{(n+1)} + \cdots + X_k^{(n+1)} | \mathcal{F}'_{n+1}) \\ &= Y^{(n+1)} \text{ a.s.} \end{aligned}$$

Hence the sequence $\{Y^{(n)}; n \geq 1\}$ is a reversed time submartingale, so as in [32] we can use the convergence theorem and the convergence in probability to conclude that

$$\liminf_{n \rightarrow \infty} X_1^{(n)} \geq \lambda \text{ a.s.} \quad (2.12)$$

Then (2.11) and (2.12) give the result.

When $\alpha_{i,r} = 1/k$ we can use the Strong Law of Large Numbers instead of Lemma 2.8 (as in the proof of Theorem 2.1(iii) of [32]) and so the final part of the statement follows. \square

The following result and its proof are based on Corollary 3.1 of [32].

Corollary 2.5. *For any set of random variables $\tilde{X}^{(n)}$ where $\tilde{X}^{(n)}$ has the same distribution as $X_1^{(n)}$ for all n ,*

$$\tilde{X}^{(n)} \rightarrow \lambda \text{ a.s.}$$

Proof. For each $n \geq 1$, $X_2^{(n)}$ is determined by the random variables $\{X_i^{(0)}; k^n + 1 \leq i \leq 2k^n\} \cup \{Z_i^{(j)}; 1 \leq j \leq n; k^{n-j} + 1 \leq i \leq 2k^{n-j}\}$. But these sets are independent for different n and so $\{X_2^{(n)}; n \geq 0\}$ is a set of independent random variables. From the proof of Lemma 2.4 we have that $X_2^{(n)} \rightarrow \lambda$ a.s. and so by Borel-Cantelli we have that

$$\sum_{n=0}^{\infty} \mathbb{P}(|X_2^{(n)} - \lambda| \geq \epsilon) < \infty \text{ for all } \epsilon > 0. \quad (2.13)$$

But, for each n , $X_2^{(n)}$ and $X_1^{(n)}$ are i.i.d. random variables. By the hypotheses of the corollary, $\tilde{X}^{(n)}$ has the same distribution as $X_2^{(n)}$ and hence (2.13) implies that

$$\sum_{n=0}^{\infty} \mathbb{P}(|\tilde{X}^{(n)} - \lambda| \geq \epsilon) < \infty \text{ for all } \epsilon > 0,$$

which, using Borel-Cantelli, gives the result. \square

We can now use this to strengthen Lemma 2.4 by removing the symmetry condition. Theorem 2.1 is a special case of the following:

Theorem 2.6. (a) Under the hypotheses of Lemma 2.3 there exists $\lambda \in \mathcal{D}$ with

$$X_1^{(n)} \rightarrow \lambda \text{ a.s.}$$

(b) If $\alpha_{i,r} = 1/k$ for all i, r then the conclusion holds under the weaker condition that $\mathbb{E}|X_1^{(0)}| < \infty$.

Proof. We define

$$g_r(x_1, \dots, x_k) = \sum_{i=1}^k \alpha_{i,r} x_i \text{ for all } (x_1, \dots, x_k) \in \mathcal{D}^k.$$

Set $\tilde{R} = R \times S_k$ where S_k is the k th permutation group. Then, for each r, σ ; $r \in R, \sigma \in S_k$, define $\tilde{f}_{(r,\sigma)} : \mathcal{D}^k \rightarrow \mathcal{D}$ by

$$\tilde{f}_{(r,\sigma)}(x_1, \dots, x_k) = f_r \circ \sigma(x_1, \dots, x_k).$$

The condition on the f_r ensures that, for all r ,

$$\begin{aligned} \tilde{f}_{(r,\sigma)}(x_1, \dots, x_k) &\leq \tilde{g}_{(r,\sigma)}(x_1, \dots, x_k) \\ &:= g_r \circ \sigma(x_1, \dots, x_k) \text{ for all } (x_1, \dots, x_k) \in \mathcal{D}^k. \end{aligned}$$

We now consider a model with random variables $Y_j^{(n)}$, based on the set of functions $\{\tilde{f}_{\tilde{r}}; \tilde{r} \in \tilde{R}\}$. The symmetry condition of Lemma 2.4 will be satisfied if we define the random variables $\tilde{Z}_i^{(n)}$ so that $\mathbb{P}(\tilde{Z}_i^{(n)} \in E \times \{\sigma\}) = (1/k!) \mathbb{P}(Z_i^{(n)} \in E)$ for measurable subsets $E \subseteq R$ and $\sigma \in S_k$. So if we set $\mathcal{Y}^{(0)} = \mathcal{X}^{(0)}$ and define i.i.d. sequences $\mathcal{Y}^{(n)} = \{Y_i^{(n)} : i \geq 1\}$ using our new model, we can apply Lemma 2.4 to obtain $Y_1^{(n)} \rightarrow \lambda$ a.s. for some $\lambda \in \mathcal{D}$.

Now note that, as $\{Y_i^{(n)} : i \geq 1\}$ are i.i.d., the distribution of $Y_i^{(1)}$ is independent of $Z_i^{(1)}$. Hence it is the same as the distribution of $X_i^{(1)}$. We continue inductively to see that $X_i^{(n)}$ and $Y_i^{(n)}$ have the same distribution for all n, i (although the joint distributions as n varies will not necessarily be the same). We now apply Corollary 2.5 to obtain the result.

To apply Lemma 2.4 we need $\mathbb{E} \left(|X_1^{(0)}| (L(|X_1^{(0)}|))^{\delta} \right) < \infty$, giving (a), except when $\alpha_i = 1/k$ when $\mathbb{E}|X_1^{(0)}| < \infty$ is sufficient, giving (b). \square

Theorem 2.1 is just Theorem 2.6 in the case $R = \{1\}$.

It is conjectured in [32] that an almost sure limit will exist for more general iterations which may vary from stage to stage, and with the initial random variables m -dependent rather than independent. In the case where k varies the reversed submartingale argument of this chapter cannot be used and the argument in Theorem 2.6 requires independence initially, so the methods used here cannot be applied in their present form to this conjecture.

2.3 Required lemmas

The following results and their proofs are extensions of Proposition 3.1 of [32] to the case with random weights.

A set of random variables $\{X_j : j \geq 1\}$ is said to be stationary and m -dependent if for all n the sequence $\{X_{n+j} : j \geq 1\}$ has the same distribution as $\{X_j : j \geq 1\}$ and the collections

$$\{X_1, \dots, X_n\} \text{ and } \{X_{n+m+1}, X_{n+m+2}, \dots\}$$

are independent.

Lemma 2.7. *Let $\{X_n; n \geq 1\}$ be a sequence of stationary and m -dependent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{a_{n,k}; k \geq 1, n \geq 1\}$ an array of non-negative real random variables on the same probability space independent of $\{X_n; n \geq 1\}$ such that, for all $\omega \in \Omega$,*

$$\sum_{k \geq 1} a_{n,k} = 1 \text{ for } n \geq 1.$$

If

$$\sup_{\omega \in \Omega} \sup_{k \geq 1} a_{n,k} \rightarrow 0 \text{ as } n \rightarrow \infty$$

and

$$\mathbb{E}(X_1 \vee 0) < \infty,$$

then for each $n \geq 1$, $\sum_{k \geq 1} a_{n,k} X_k$ is a well-defined $[-\infty, \infty)$ -valued random variable with

$$\mathbb{E} \left(\sum_{k \geq 1} a_{n,k} X_k \right) = \mathbb{E} X_1$$

and

$$\sum_{k \geq 1} a_{n,k} X_k \rightarrow_p \mathbb{E} X_1.$$

(with convergence to $-\infty$ if $\mathbb{E} X_1^{(0)} = -\infty$)

Proof. We define the σ -algebra $\mathcal{G}_n = \sigma(a_{n,k}; k \geq 1)$. By conditioning on \mathcal{G}_n it is obvious that

$$\mathbb{E} \left(\sum_{k \geq 1} a_{n,k} X_k \right) = \mathbb{E} X_1.$$

For the second part, first assume that $\mathbb{E}(X_1 \wedge 0) > -\infty$. Then we know that

$$\mathbb{E}|X_1| < \infty$$

from the hypothesis that $\mathbb{E}(X_1 \vee 0) < \infty$. Given $\epsilon > 0$ we choose $\tau > 0$ such that

$$\mathbb{E}(|X_1| I_{\{|X_1| \geq \tau\}}) \leq \frac{\epsilon^2}{2}.$$

Now, for $j \geq 1$, define the following two random variables:

$$Y_j(\tau) = X_j I_{\{|X_j| \leq \tau\}} - \mathbb{E}(X_1 I_{\{|X_1| \leq \tau\}}),$$

$$Z_j(\tau) = X_j - \mathbb{E} X_1 - Y_j(\tau).$$

Then because $\sum_{k \geq 1} a_{n,k} = 1$ for all $\omega \in \Omega$ and all $n \geq 1$, we have from the definition of $Z_j(\tau)$,

$$\sum_{k \geq 1} a_{n,k} X_k - \mathbb{E} X_1 = \sum_{k \geq 1} a_{n,k} Y_k(\tau) + \sum_{k \geq 1} a_{n,k} Z_k(\tau).$$

Now

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{k \geq 1} a_{n,k} Z_k(\tau) \right| \geq \epsilon \middle| \mathcal{G}_n \right) &\leq \frac{1}{\epsilon} \mathbb{E} \left(\left| \sum_{k \geq 1} a_{n,k} Z_k(\tau) \right| \middle| \mathcal{G}_n \right) \\ &\leq \frac{1}{\epsilon} \sum_{k \geq 1} a_{n,k} \mathbb{E} (|Z_k(\tau)| \middle| \mathcal{G}_n). \end{aligned}$$

Now $\mathbb{E}|Z_k(\tau)| < 2\mathbb{E}(|X_1|I_{\{|X_1| \geq \tau\}})$ by the definition of $Z_k(\tau)$ so we use properties of conditional expectation to get

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{k \geq 1} a_{n,k} Z_k(\tau) \right| \geq \epsilon \right) &\leq \frac{2}{\epsilon} \mathbb{E} (|X_1|I_{\{|X_1| > \tau\}}) \\ &\leq \epsilon. \end{aligned}$$

We now use the stationarity and m -dependence of $\{X_n; n \geq 1\}$ to get

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{k \geq 1} a_{n,k} Y_k(\tau) \right| \geq \epsilon \middle| \mathcal{G}_n \right) &\leq \frac{1}{\epsilon^2} \text{Var} \left(\sum_{k \geq 1} a_{n,k} Y_k(\tau) \middle| \mathcal{G}_n \right) \\ &= \frac{1}{\epsilon^2} \left(\sum_{k \geq 1} a_{n,k}^2 \text{Var}(Y_k(\tau) | \mathcal{G}_n) \right. \\ &\quad \left. + 2 \sum_{1 \leq i-j \leq m} a_{n,i} a_{n,j} \text{Cov}(Y_i(\tau), Y_j(\tau) | \mathcal{G}_n) \right) \\ &\leq \frac{2}{\epsilon^2} (m+1) \sum_{k \geq 1} a_{n,k}^2 \text{Var}(Y_k(\tau) | \mathcal{G}_n) \\ &\leq \frac{2(m+1)\tau^2}{\epsilon^2} \sup_{k \geq 1} a_{n,k} \sum_{k \geq 1} a_{n,k} \\ &= \frac{2(m+1)\tau^2}{\epsilon^2} \sup_{k \geq 1} a_{n,k}. \end{aligned}$$

Now remove the conditioning and use the condition on the $a_{n,k}$ to obtain

$$\mathbb{P} \left(\left| \sum_{k \geq 1} a_{n,k} Y_k(\tau) \right| \geq \epsilon \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence

$$\limsup \mathbb{P} \left(\left| \sum_{k \geq 1} a_{n,k} X_k - \mathbb{E} X_1 \right| \geq 2\epsilon \right) \leq \epsilon,$$

from which we can deduce the result, when $\mathbb{E}(X_1 \wedge 0) > -\infty$. If $\mathbb{E}(X_1 \wedge 0) = -\infty$, for each $M > 0$, we truncate and apply the previous argument to $X_k I_{\{X_k \geq -M\}}$ as in [32]. \square

Lemma 2.8. *With the same framework as Lemma 2.7, and further that*

$$\zeta_n^{-1} := \sum_{k \geq 1} a_{n,k}^2 \leq cb^{-n} \text{ for all } n \geq 1, \omega \in \Omega$$

for some constants $b > 1$ and $c > 0$, and

$$\mathbb{E}(|X_1|(L(|X_1|))^\delta) < \infty$$

for some $\delta > 1$, where $L(x) := \log(\max\{e, x\})$, then

$$\sum_{n \geq 1} \mathbb{P} \left(\left| \sum_{k \geq 1} a_{n,k} X_k - \mathbb{E} X_1 \right| \geq \epsilon \right) < \infty, \text{ for all } \epsilon > 0$$

and hence (via Borel-Cantelli) we have that

$$\lim_{n \rightarrow \infty} \sum_{k \geq 1} a_{n,k} X_k = \mathbb{E} X_1 \text{ a.s.}$$

Proof. By subtracting $\mathbb{E} X_1$ from the initial random variables, we can assume $\mathbb{E} X_1 = 0$.

For each $n \geq 1$, define the following random variables:

$$U_n = \sum_{k \geq 1} a_{n,k} X_k I_{\{|X_k| \leq \zeta_n\}},$$

$$V_n = \sum_{k \geq 1} a_{n,k} X_k I_{\{|X_k| > \zeta_n\}},$$

and define $\mathcal{G} = \sigma(a_{n,k}; n \geq 1, k \geq 1)$. Now

$$\begin{aligned}
\sum_{n \geq 1} \mathbb{P}(|V_n - \mathbb{E}(V_n | \mathcal{G})| \geq \epsilon | \mathcal{G}) &\leq \frac{1}{\epsilon} \sum_{n \geq 1} \mathbb{E}(|V_n - \mathbb{E}(V_n | \mathcal{G})| | \mathcal{G}) \\
&= \frac{1}{\epsilon} \sum_{n \geq 1} \mathbb{E}(|X_1 I_{\{|X_1| > \zeta_n\}} - \mathbb{E}(X_1 I_{\{|X_1| > \zeta_n\}} | \mathcal{G})| | \mathcal{G}) \\
&\quad \text{by the definition of } V_n. \\
&\leq \frac{2}{\epsilon} \sum_{n \geq 1} \mathbb{E}(|X_1 I_{\{|X_1| > \zeta_n\}}| | \mathcal{G}) \\
&\quad \text{by the triangle and Jensen's inequalities.}
\end{aligned}$$

Now remove the conditioning to get

$$\begin{aligned}
\sum_{n \geq 1} \mathbb{P}(|V_n - \mathbb{E}V_n| \geq \epsilon) &\leq \frac{2}{\epsilon} \sum_{n \geq 1} \mathbb{E}(|X_1 I_{\{|X_1| > b^n/c\}}|) \\
&\leq \frac{2}{\epsilon} \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \frac{b^{j+1}}{c} \mathbb{P}(b^j/c < |X_1| < b^{j+1}/c) \\
&\leq \frac{2}{\epsilon} \sum_{j=1}^{\infty} \frac{jb^{j+1}}{c} \mathbb{P}(b^j < |cX_1| < b^{j+1}) \\
&\leq \frac{2b}{\epsilon \log b} \mathbb{E}(|X_1| L(|cX_1|)) \\
&< \infty.
\end{aligned}$$

Now we use variance and covariance arguments again to get

$$\begin{aligned}
\mathbb{P}(|U_n - \mathbb{E}(U_n | \mathcal{G})| \geq \epsilon | \mathcal{G}) &\leq \frac{1}{\epsilon^2} \text{Var}(U_n | \mathcal{G}) \\
&\leq \frac{m+1}{\epsilon^2} \sum_{k \geq 1} a_{n,k}^2 \text{Var}(X_1 I_{\{|X_1| \leq \zeta_n\}} | \mathcal{G}) \\
&\leq \frac{m+1}{\epsilon^2} \zeta_n^{-1} \text{Var}(X_1 I_{\{|X_1| \leq \zeta_n\}} | \mathcal{G}).
\end{aligned}$$

Now remove the conditioning to get

$$\begin{aligned}
\sum_{n \geq 1} \mathbb{P}(|U_n - \mathbb{E}U_n| \geq \epsilon) &\leq \frac{m+1}{\epsilon^2} \sum_{n \geq 1} cb^{-n} \text{Var}(X_1 I_{\{|X_1| \leq b^n/c\}}) \\
&\leq \frac{m+1}{\epsilon^2} \sum_{n \geq 1} cb^{-n} \mathbb{E}(X_1^2 I_{\{|X_1| \leq b^n/c\}}) \\
&= \frac{m+1}{\epsilon^2} \sum_{n \geq 1} cb^{-n} \left(\frac{b^n}{c} O\left(\frac{1}{(L(\frac{b^n}{c}))^\delta}\right) \mathbb{E}(|X_1|(L(|X_1|))^\delta) \right) \\
&= \sum_{n \geq 1} O\left(\frac{1}{n^\delta}\right) \\
&< \infty.
\end{aligned}$$

Combining the results for U_n and V_n we can now use the Borel-Cantelli Lemma to get the result. \square

Chapter 3

Random fractal resistor networks

3.1 Introduction

Many properties of non-homogeneous random fractal graphs can be studied using random hierarchical systems. One example is obtained by considering the graph as an electrical network where the edges are independent and identically distributed random resistors.

As an example network we consider the non-homogeneous constructions where Γ_1 consists of a two-edge path and Γ_2 consists of a pair of vertices connected by two edges, as shown in Figure 3.1.

We set a parameter $p \in [0, 1]$ and take the random variables $Z_j^{(n)}$ to be 1 with probability p and 2 with probability $1 - p$.

We will call this the *series-parallel* network with parameter p .

Considering the replacement construction, we now put a resistance of 1 on each edge and define $R^{(n)}$ to be the resistance between the boundary points

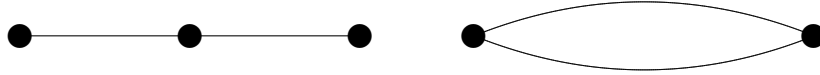


Figure 3.1: The model graphs for the series-parallel network

in the graph $G^{(n)}$. A typical graph at stage 5 with $p = \frac{1}{2}$ might look like that in Figure 3.2.

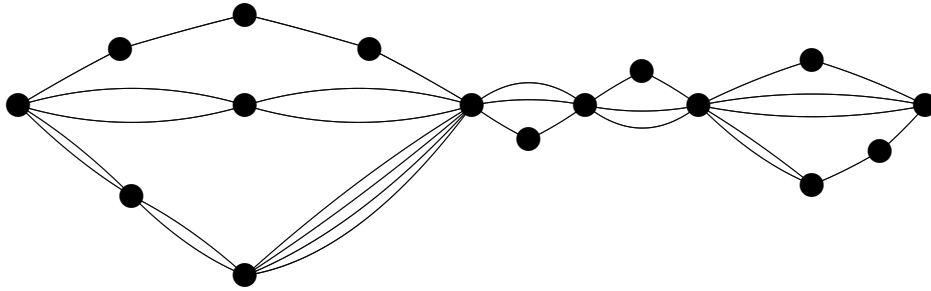


Figure 3.2: An sample series-parallel graph, $G^{(5)}$

To investigate the resistance of this network, we consider a random hierarchical system $\{X_j^{(n)}; n, j \in \mathbb{N}\}$ with $R = \{1, 2\}$ and the two functions

$$\begin{aligned} f_1(x_1, x_2) &= x_1 + x_2 \\ f_2(x_1, x_2) &= \frac{x_1 x_2}{x_1 + x_2}. \end{aligned}$$

with the measure μ_p giving probability p on 1 and probability $1 - p$ on 2.

We set $X_j^{(0)} = 1$ for all j . Then $X_j^{(n)}$ is the resistance of the duplication graph $G_j^{(n)}$.

Lemma 1.5 tells us that the random variables $R^{(n)}$ and $X_j^{(n)}$ have the same distribution for all $n, j \in \mathbb{N}$. Note however that the distribution of the sequences $(R^{(n)})_{n \in \mathbb{N}}$ and $(X_1^{(n)})_{n \in \mathbb{N}}$ is not the same.

For a random fractal graph with the number of boundary points $M = 2$, we can define the following processes:

Definition 3.1. *Set $C^{(n)}$ to be the number of individual edges whose removal would disconnect $G^{(n)}$. This behaves as a Galton-Watson branching process with offspring distribution corresponding to the distribution of the number of such edges in the model graphs.*

Similarly set $\tilde{C}^{(n)}$ to be the number $w^{(n)}(1, 2)$ of edges connecting the two boundary points of the network. This also behaves as a branching process.

Lemma 3.2. *For the series-parallel network with $p > \frac{1}{2}$, there is a positive probability of $R^{(n)} \rightarrow \infty$.*

Proof. The branching process $C^{(n)}$ is a lower bound for the resistance $R^{(n)}$ (as each edge has resistance 1) and its offspring mean is $2p + 0(1 - p)$, so it is supercritical for $p > \frac{1}{2}$. \square

This also shows a positive probability of a growth rate of at least $2p$ in this case.

For the series-parallel network, the offspring mean of $(\tilde{C}^{(n)})_{n \in \mathbb{N}}$ is $2(1 - p)$, so it is supercritical for $p < \frac{1}{2}$.

Although these branching processes can be defined for any such sequence of graphs, one or both will often be trivial.

3.1.1 A zero-one law

Note that, for $n > 1$, the graph $G^{(n)}$ consists of k independent graphs $G_1^{(n-1)}, \dots, G_k^{(n-1)}$ (where $G_j^{(n-1)}$ has resistance $R_j^{(n-1)}$), each equal in distri-

bution to $G^{(n-1)}$ connected together in a manner determined independently by $G^{(1)}$.

For the series-parallel network, consider an event A determined by the sequence $(G^{(n)})_{n \in \mathbb{N}}$ such that if $G^{(1)}$ has a series connection $A = A_1 \cup A_2$ (where A_1 and A_2 are the equivalent events for the sequences $(G_1^{(n)})_{n \in \mathbb{N}}$ and $(G_2^{(n)})_{n \in \mathbb{N}}$), and that if $G^{(1)}$ has a parallel connection then $A = A_1 \cap A_2$.

Lemma 3.3. *If $p \neq \frac{1}{2}$, then any such event A must have probability 0 or 1. This also applies if \cap and \cup are reversed in the above.*

Proof. Let $q = \mathbb{P}(A)$. Then q satisfies

$$q = h(q) = p(1 - (1 - q)^2) + (1 - p)q^2 = (1 - 2p)q^2 + 2pq$$

which has roots 0 and 1 unless $p = \frac{1}{2}$ when it reduces to $q = q$.

The second statement follows by the same method. □

Corollary 3.4. *We have $R_1^{(n)} \rightarrow \infty$ almost surely for $p > \frac{1}{2}$, and furthermore $2p$ as a lower bound on the exponential growth rate.*

Proof. Both the events $\{R^{(n)} \rightarrow \infty\}$ and $\{\liminf_{n \rightarrow \infty} (R^{(n)})^{\frac{1}{n}} > 2p\}$ satisfy the conditions of Lemma 3.3. As we have already shown that they have positive probability, they must have probability 1. □

Similarly we get

Corollary 3.5. *We have $R^{(n)} \rightarrow 0$ almost surely for $p < \frac{1}{2}$, with exponential decay.*

However the branching process argument does not help at $p = \frac{1}{2}$. Also note that because the sequences are different, almost sure convergence of $R^{(n)}$ does not imply almost sure convergence of $X_1^{(n)}$. This will be proved in section 3.2.

A polynomial $h(q)$ corresponding to $(1 - 2p)q^2 + 2pq$ can be constructed for other networks. In the case where there is just one function (e.g. a deterministic fractal lattice) then the polynomial $h(q)$ is studied in [38, 39], where it is shown that it has at most one fixed point in $(0, 1)$ which will be repulsive. It is also studied in [47]. These results do not apply to randomised networks, but the fixed points of $h(q)$ still provide useful information.

Also note that $h'(0)$ is equal to the offspring mean of the branching process $C^{(n)}$ from Definition 3.1 and that $h'(1)$ is equal to the offspring mean of the branching process $\tilde{C}^{(n)}$.

We will consider the equivalent results for networks based on model graphs with 3 edges in section 3.3.

3.2 Asymptotics for the series-parallel network

We now consider the series-parallel network from the last section and investigate how the asymptotics depend on the parameter p . We will show that for $p < \frac{1}{2}$ we have $X_1^{(n)} \rightarrow 0$ almost surely and that for $p > \frac{1}{2}$ we have $X_1^{(n)} \rightarrow \infty$ almost surely, and that in both cases the convergence is exponentially fast. When $p = \frac{1}{2}$ we will show that exponential growth and decay have probability 0, but show that $\mathbb{E}(\log X_1^{(n)})^2 \rightarrow \infty$. This suggests that there may not be a non-degenerate fixed point.

3.2.1 Symmetry properties

Lemma 3.6. *The distribution of $X_j^{(n)}$ with parameter p is the same as the distribution of $1/X_j^{(n)}$ with parameter $1 - p$.*

Proof. Given $X_j^{(n)}$ we construct $\tilde{X}_j^{(n)}$ as follows:

We set $\tilde{X}_j^{(0)} = X_j^{(0)} = 1$.

If $X_j^{(n)} = f_1(X_{2j-1}^{(n-1)}, X_{2j}^{(n-1)})$ then $\tilde{X}_j^{(n)} = f_2(\tilde{X}_{2j-1}^{(n-1)}, \tilde{X}_{2j}^{(n-1)})$.

If $X_j^{(n)} = f_2(X_{2j-1}^{(n-1)}, X_{2j}^{(n-1)})$ then $\tilde{X}_j^{(n)} = f_1(\tilde{X}_{2j-1}^{(n-1)}, \tilde{X}_{2j}^{(n-1)})$.

Then $\tilde{X}_j^{(n)}$ with parameter p is distributed as $X_j^{(n)}$ with parameter $1 - p$.

But $\tilde{X}_j^{(0)} = 1/X_j^{(0)}$ for all j . Assuming (induction hypothesis), that $\tilde{X}_l^{(n-1)} = 1/X_l^{(n-1)}$ for all l then if

$$X_j^{(n)} = f_1(X_{2j-1}^{(n-1)}, X_{2j}^{(n-1)}),$$

then

$$\tilde{X}_j^{(n)} = \frac{1}{\frac{1}{\tilde{X}_{2j-1}^{(n-1)}} + \frac{1}{\tilde{X}_{2j}^{(n-1)}}} = \frac{1}{X_{2j-1}^{(n-1)} + X_{2j}^{(n-1)}} = \frac{1}{X_j^{(n)}}$$

and if

$$X_j^{(n)} = f_2(X_{2j-1}^{(n-1)}, X_{2j}^{(n-1)}),$$

then

$$\tilde{X}_j^{(n)} = \tilde{X}_{2j-1}^{(n-1)} + \tilde{X}_{2j}^{(n-1)} = \frac{1}{X_{2j-1}^{(n-1)}} + \frac{1}{X_{2j}^{(n-1)}} = \frac{1}{X_j^{(n)}}.$$

So $\tilde{X}_j^{(n)} = 1/X_j^{(n)}$ for all k □

Corollary 3.7. *When $p = \frac{1}{2}$, the distribution of $Y_1^{(n)} = \log X_1^{(n)}$ is symmetric about 0 for all n .*

We will say that $X_j^{(n)}$ is *log-symmetric* in this case.

3.2.2 Behaviour of $\log X_1^{(n)}$

We set $Y_j^{(n)} = \log X_j^{(n)}$ and $q = 1 - p$.

Now note that

$$\begin{aligned} \mathbb{E}(Y_j^{(n)} | \mathcal{F}_{n-1}) &= p \log(X_{2j-1}^{(n-1)} + X_{2j}^{(n-1)}) + q \log\left(\frac{X_{2j-1}^{(n-1)} X_{2j}^{(n-1)}}{X_{2j-1}^{(n-1)} + X_{2j}^{(n-1)}}\right) \\ &= (p - q) \log(X_{2j-1}^{(n-1)} + X_{2j}^{(n-1)}) + q(Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}) \end{aligned}$$

When $p = q = \frac{1}{2}$, this simplifies to

$$\mathbb{E}(Y_j^{(n)} | \mathcal{F}_{n-1}) = \frac{1}{2}(Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}), \quad (3.1)$$

a property reminiscent of the martingale property. Because we are assuming that the $X_j^{(n)}$ are i.i.d. for fixed n this gives

$$\mathbb{E}(Y_1^{(n)}) = \mathbb{E}(Y_1^{(n-1)}) = \mathbb{E}(Y_1^{(0)}) = 0.$$

Jensen's inequality turns this into

$$\mathbb{E}(X_j^{(n)}) \geq e^{\mathbb{E}(Y_j^{(n)})} = 1.$$

When $p \neq \frac{1}{2}$ we get

$$\mathbb{E}(Y_1^{(n)}) = 2q\mathbb{E}(Y_1^{(n-1)}) + (p - q)\mathbb{E}(\log(X_1^{(n-1)} + X_2^{(n-1)})).$$

Now $1 - 2q = (p + q) - 2q = p - q$, so this is

$$\begin{aligned} \mathbb{E}(Y_1^{(n)}) &= \mathbb{E}(Y_1^{(n-1)}) + (p - q)\mathbb{E}(\log(X_1^{(n-1)} + X_2^{(n-1)}) - \log X_1^{(n-1)}) \\ &= \mathbb{E}(Y_1^{(n-1)}) + (p - q)\mathbb{E}\left(\log\left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{X_1^{(n-1)}}\right)\right). \end{aligned}$$

Lemma 3.8. *If X_1 and X_2 are independent and identically distributed positive random variables, then*

$$\mathbb{E} \log \left(\frac{X_1 + X_2}{X_1} \right) \geq \log 2.$$

Proof. Let μ be the distribution of X_1 . Then

$$\begin{aligned} \mathbb{E} \log \left(\frac{X_1 + X_2}{X_1} \right) &= \mathbb{E} \left(\log \left(\frac{X_1 + X_2}{X_1} \right) 1_{\{X_2 > X_1\}} \right) \\ &\quad + \mathbb{E} \left(\log \left(\frac{X_1 + X_2}{X_1} \right) 1_{\{X_2 < X_1\}} \right) + \mathbb{E} \left(\log \left(\frac{X_1 + X_2}{X_1} \right) 1_{\{X_2 = X_1\}} \right) \end{aligned}$$

$$\begin{aligned}
&= \int_{x_2 > x_1} \log\left(\frac{x_1 + x_2}{x_1}\right) d\mu(x_1)d\mu(x_2) + \int_{x_2 < x_1} \log\left(\frac{x_1 + x_2}{x_2}\right) d\mu(x_1)d\mu(x_2) \\
&\quad + \log 2 \int_{x_1 = x_2} d\mu(x_1)d\mu(x_2) \\
&= \int_{x_2 > x_1} \log\left(\frac{x_1 + x_2}{2x_1}\right) d\mu(x_1)d\mu(x_2) \\
&\quad - \int_{x_2 < x_1} \log\left(\frac{2x_2}{x_1 + x_2}\right) d\mu(x_1)d\mu(x_2) + \log 2 \\
&= \int_{x_2 > x_1} \log\left(\frac{x_1 + x_2}{2x_1}\right) d\mu(x_1)d\mu(x_2) \\
&\quad - \int_{x_2 > x_1} \log\left(\frac{2x_1}{x_1 + x_2}\right) d\mu(x_1)d\mu(x_2) + \log 2
\end{aligned}$$

by swapping variables in the second integral. This then becomes

$$\begin{aligned}
\mathbb{E} \log\left(\frac{X_1 + X_2}{X_1}\right) &= \log 2 + \int_{x_2 > x_1} \log\left(\frac{(x_1 + x_2)^2}{4x_1^2}\right) d\mu(x_1)d\mu(x_2) \\
&\geq \log 2
\end{aligned}$$

□

Corollary 3.9. *For $p \leq \frac{1}{2}$, we have*

$$\mathbb{E}(Y_1^{(n)}) = \mathbb{E} \log(X_1^{(n)}) \leq -(q - p)n \log 2$$

and for $p \geq \frac{1}{2}$, we have

$$\mathbb{E}(Y_1^{(n)}) = \mathbb{E} \log(X_1^{(n)}) \geq (p - q)n \log 2.$$

In the latter case, Jensen's inequality implies that

$$\mathbb{E}(X_1^{(n)}) \geq 2^{n(p-q)}.$$

Proof. This is immediate from the lemma and the previous work. □

Given the martingale-type property (3.1) in the $p = \frac{1}{2}$ case, we can look at the variance of $Y_1^{(n)} = \log X_1^{(n)}$ as follows:

$$\begin{aligned} \mathbb{E} \left(\left(\frac{Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}}{2} \right) \left(Y_j^{(n)} - \frac{Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}}{2} \right) \middle| \mathcal{F}_{n-1} \right) \\ = \left(\frac{Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}}{2} \right) \mathbb{E} \left(Y_j^{(n)} - \frac{Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}}{2} \middle| \mathcal{F}_{n-1} \right). \end{aligned}$$

But the right hand side is zero, so $Y_j^{(n)} - \frac{Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}}{2}$ and $\frac{Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}}{2}$ are uncorrelated.

Hence

$$\text{Var } Y_j^{(n)} = \text{Var} \left(\frac{Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}}{2} \right) + \text{Var} \left(Y_j^{(n)} - \frac{Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}}{2} \right) \quad (3.2)$$

$$= \frac{1}{2} \text{Var } Y_j^{(n-1)} + \text{Var} \left(Y_j^{(n)} - \frac{Y_{2j-1}^{(n-1)} + Y_{2j}^{(n-1)}}{2} \right). \quad (3.3)$$

Theorem 3.10. *If $p = \frac{1}{2}$ then $\text{Var } Y_1^{(n)}$ is a strictly increasing sequence.*

Proof.

$$\begin{aligned} \text{Var} \left(Y_1^{(n)} - \frac{Y_1^{(n-1)} + Y_2^{(n-1)}}{2} \right) &= \mathbb{E} \left(\log \frac{X_1^{(n)}}{\sqrt{X_1^{(n-1)} X_2^{(n-1)}}} \right)^2 \\ &= \frac{1}{2} \left[\int \left(\log \frac{x_1 + x_2}{\sqrt{x_1 x_2}} \right)^2 d\mu_{n-1}(x_1) d\mu_{n-1}(x_2) + \int \left(\log \frac{\sqrt{x_1 x_2}}{x_1 + x_2} \right)^2 d\mu_{n-1}(x_1) d\mu_{n-1}(x_2) \right] \\ &= \int \left(\log \frac{x_1 + x_2}{\sqrt{x_1 x_2}} \right)^2 d\mu_{n-1}(x_1) d\mu_{n-1}(x_2). \end{aligned}$$

So

$$\mathbb{E}(Y_1^{(n)})^2 = \frac{1}{2} (Y_1^{(n-1)})^2 + \mathbb{E} \left(\log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{\sqrt{X_1^{(n-1)} X_2^{(n-1)}}} \right) \right)^2.$$

Now

$$\begin{aligned} \left(\log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{\sqrt{X_1^{(n-1)} X_2^{(n-1)}}} \right) \right)^2 &= (\log(X_1^{(n-1)} + X_2^{(n-1)}))^2 + \frac{1}{4} (\log X_1^{(n-1)} X_2^{(n-1)})^2 \\ &\quad - \log(X_1^{(n-1)} + X_2^{(n-1)}) \log(X_1^{(n-1)} X_2^{(n-1)}) \end{aligned}$$

We now take expectations, using the facts that $\mathbb{E} \log X_1^{(n-1)} = 0$ and that $X_1^{(n-1)}$ and $X_2^{(n-1)}$ are independent, getting

$$\begin{aligned} \mathbb{E} \left(\log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{\sqrt{X_1^{(n-1)} X_2^{(n-1)}}} \right) \right)^2 &= \mathbb{E} (\log(X_1^{(n-1)} + X_2^{(n-1)}))^2 + \frac{1}{4} \mathbb{E} (\log X_1^{(n-1)})^2 + \\ &\quad \frac{1}{4} (\log X_2^{(n-1)})^2 + \frac{1}{2} \mathbb{E} \log X_1^{(n-1)} \mathbb{E} \log X_2^{(n-1)} - \mathbb{E} (\log(X_1^{(n-1)} + X_2^{(n-1)}) \log(X_1^{(n-1)} X_2^{(n-1)})) \end{aligned}$$

so

$$\begin{aligned} \mathbb{E} \left(\log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{\sqrt{X_1^{(n-1)} X_2^{(n-1)}}} \right) \right)^2 &= \mathbb{E} \left(\log(X_1^{(n-1)} + X_2^{(n-1)}) \log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{X_1^{(n-1)} X_2^{(n-1)}} \right) \right) \\ &\quad + \frac{1}{2} \text{Var } Y_1^{(n-1)}. \end{aligned}$$

So

$$\mathbb{E} (Y_1^{(n)})^2 = \mathbb{E} (Y_1^{(n-1)})^2 + \mathbb{E} \left(\log(X_1^{(n-1)} + X_2^{(n-1)}) \log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{X_1^{(n-1)} X_2^{(n-1)}} \right) \right).$$

Now

$$\begin{aligned} &\mathbb{E} \left(\log(X_1^{(n-1)} + X_2^{(n-1)}) \log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{X_1^{(n-1)} X_2^{(n-1)}} \right) \right) \\ &= \mathbb{E} \left(\log(X_1^{(n-1)} + X_2^{(n-1)}) \log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{X_1^{(n-1)} X_2^{(n-1)}} \right) \mathbf{1}_{\{X_2^{(n-1)} > 1\}} \right) \\ &\quad + \mathbb{E} \left(\log \left(X_1^{(n-1)} + \frac{1}{X_2^{(n-1)}} \right) \log \left(\frac{X_1^{(n-1)} + \frac{1}{X_2^{(n-1)}}}{X_1^{(n-1)} \frac{1}{X_2^{(n-1)}}} \right) \mathbf{1}_{\{X_2^{(n-1)} > 1\}} \right) \\ &\quad + \mathbb{E} \left(\log(X_1^{(n-1)} + X_2^{(n-1)}) \log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{X_1^{(n-1)} X_2^{(n-1)}} \right) \mathbf{1}_{\{X_2^{(n-1)} = 1\}} \right) \end{aligned}$$

using independence of $X_1^{(n-1)}$ and $X_2^{(n-1)}$ plus the symmetry of the distributions of the logs:

$$X_2^{(n-1)} =_d \frac{1}{X_2^{(n-1)}}.$$

So

$$\begin{aligned} & \mathbb{E} \left(\log(X_1^{(n-1)} + X_2^{(n-1)}) \log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{X_1^{(n-1)} X_2^{(n-1)}} \right) \right) \\ & \geq \mathbb{E} \left(\left(\log(X_1^{(n-1)} + X_2^{(n-1)}) \log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{X_1^{(n-1)} X_2^{(n-1)}} \right) \right. \right. \\ & \quad \left. \left. + \log \left(\frac{X_1^{(n-1)} X_2^{(n-1)} + 1}{X_2^{(n-1)}} \right) \log \left(\frac{X_1^{(n-1)} X_2^{(n-1)} + 1}{X_1^{(n-1)}} \right) \right) 1_{\{X_2^{(n-1)} > 1\}} \right). \end{aligned}$$

So we look at

$$v(x_1, x_2) = \log(x_1 + x_2) \log \left(\frac{x_1 + x_2}{x_1 x_2} \right) + \log \left(\frac{x_1 x_2 + 1}{x_2} \right) \log \left(\frac{x_1 x_2 + 1}{x_1} \right).$$

Expanding gives

$$\begin{aligned}
v(x_1, x_2) &= (\log(x_1 + x_2))^2 - \log(x_1 x_2) \log(x_1 + x_2) + (\log(x_1 x_2 + 1))^2 \\
&\quad + (\log x_1)(\log x_2) - (\log(x_1 x_2 + 1) \log(x_1 x_2)) \\
&= (\log(x_1 + x_2))^2 + (\log(x_1 x_2 + 1)(\log(x_1 x_2 + 1) - \log(x_1 x_2)) \\
&\quad + (\log x_1)(\log x_2) - \log(x_1 x_2) \log(x_1 + x_2)) \\
&\geq (\log(x_1 + x_2))^2 + \log x_1 \log x_2 - \log(x_1 x_2) \log(x_1 + x_2) \\
&= (\log(x_1 + x_2))^2 + \log x_1 \log x_2 - \log x_1 \log(x_1 + x_2) \\
&\quad - \log x_2 \log(x_1 + x_2) \\
&= (\log(x_1 + x_2))^2 + \log x_1 \log x_2 - \frac{1}{2}(\log x_1)^2 - \frac{1}{2}(\log(x_1 + x_2))^2 \\
&\quad - \frac{1}{2}(\log x_2)^2 - \frac{1}{2}(\log(x_1 + x_2))^2 + \frac{1}{2}(\log\left(\frac{x_1}{x_1 + x_2}\right))^2 \\
&\quad + \frac{1}{2}(\log\left(\frac{x_2}{x_1 + x_2}\right))^2 \\
&= \frac{1}{2}(\log\left(\frac{x_1}{x_1 + x_2}\right))^2 + \frac{1}{2}(\log\left(\frac{x_2}{x_1 + x_2}\right))^2 - (\log x_1 - \log x_2)^2 \\
&= \frac{1}{2}(\log\left(\frac{x_1}{x_1 + x_2}\right))^2 + \frac{1}{2}(\log\left(\frac{x_2}{x_1 + x_2}\right))^2 - \left(\log\frac{x_1}{x_1 + x_2} \right. \\
&\quad \left. - \log\frac{x_2}{x_1 + x_2}\right)^2 \\
&= \left(\log\frac{x_1}{x_1 + x_2}\right) \left(\log\frac{x_2}{x_1 + x_2}\right) > 0 \text{ if } x_1, x_2 > 0.
\end{aligned}$$

This means that, for the case $p = \frac{1}{2}$, if the distribution of $\log X_1^{(n-1)}$ is symmetric about 0, we have

$$\mathbb{E}(\log X_1^{(n)})^2 > \mathbb{E}(\log X_1^{(n-1)})^2.$$

□

Corollary 3.11. *In the $p = \frac{1}{2}$ case, if there exists a symmetric distribution μ_∞ (in the sense that a random variable X with distribution μ_∞ has $X \stackrel{d}{=} \frac{1}{X}$) which is a fixed point of the underlying map, then if X has distribution μ_∞ , then $\mathbb{E}(\log X)^2$ cannot exist.*

3.2.3 Limiting behaviour for $p \neq \frac{1}{2}$

We can use recurrences for expectation to prove convergence to zero in some cases, almost surely. The simplest version uses the following lemma.

Lemma 3.12. *For all p , $\mathbb{E}X_1^{(n)} \leq \left(\frac{3p+1}{2}\right)^n$.*

Proof. We know that

$$\mathbb{E}(X_j^{(n)} | \mathcal{F}_{n-1}) = p(X_{2j-1}^{(n-1)} + X_{2j}^{(n-1)}) + (1-p) \frac{X_{2j-1}^{(n-1)} X_{2j}^{(n-1)}}{X_{2j-1}^{(n-1)} + X_{2j}^{(n-1)}}.$$

Using the inequality of arithmetic and harmonic means, this gives

$$\mathbb{E}(X_j^{(n)} | \mathcal{F}_{n-1}) \leq \left(p + \frac{1-p}{4}\right) (X_{2j-1}^{(n-1)} + X_{2j}^{(n-1)})$$

Now $(p + \frac{1-p}{4}) = \frac{1}{2} \left(\frac{3p+1}{2}\right)$ and we can remove the conditional expectation to get

$$\mathbb{E}X_1^{(n)} \leq \left(\frac{3p+1}{2}\right) \mathbb{E}X_1^{(n-1)}$$

using the i.i.d property of $X_j^{(n)}$ for fixed n . This immediately gives the result. \square

Corollary 3.13. *For $p < \frac{1}{3}$, the sequence $(X_1^{(n)})_{n \in \mathbb{N}}$ converges almost surely to zero.*

Proof. From above we have

$$\mathbb{E}|X_1^{(n)}| \leq \left(\frac{3p+1}{2}\right)^n$$

which using Markov's inequality gives

$$\mathbb{P}\{X_1^{(n)} \geq \left(\frac{3p+1}{2}\right)^m\} \leq \left(\frac{3p+1}{2}\right)^{n-m}$$

Because $\frac{3p+1}{2} < 1$ for $p < \frac{1}{3}$ this gives convergence to zero exponentially fast in probability. Because

$$\sum_{n=0}^{\infty} \left(\frac{3p+1}{2} \right)^{n-m} < \infty$$

we can use the Borel-Cantelli Lemma to conclude that $X_1^{(n)}$ converges to zero almost surely. \square

Corollary 3.14. *If $p > \frac{2}{3}$ then $X_1^{(n)}$ converges almost surely to ∞ .*

Proof. This is a consequence of Lemma 3.6 and Corollary 3.13. \square

To extend this to the interval $[\frac{1}{3}, \frac{1}{2})$ we take an integer α and consider the behaviour of

$$\sqrt[\alpha]{X_j^{(n)}}.$$

Now

$$\mathbb{E}(\sqrt[\alpha]{X_j^{(n)}} | \mathcal{F}_{n-1}) = p(X_{2j-1}^{(n-1)} + X_{2j}^{(n-1)})^{\frac{1}{\alpha}} + (1-p) \left(\frac{X_{2j-1}^{(n-1)} X_{2j}^{(n-1)}}{X_{2j-1}^{(n-1)} + X_{2j}^{(n-1)}} \right)^{\frac{1}{\alpha}}.$$

So if

$$p(1+x)^{\frac{1}{\alpha}} + (1-p) \left(\frac{x}{1+x} \right)^{\frac{1}{\alpha}} \leq \frac{C}{2}(1+x^{\frac{1}{\alpha}})$$

for some C (depending on α) and for all x , then

$$\mathbb{E} \sqrt[\alpha]{X_j^{(n)}} \leq C^n,$$

so if $C < 1$ then $\sqrt[\alpha]{X_1^{(n)}} \rightarrow 0$ almost surely by the argument used in Lemma 3.13 for the $p < \frac{1}{3}$ case, which corresponds to $\alpha = 1$.

Hence we are interested in the supremum over x of

$$f(x, \beta, p) = \frac{p(1+x)^\beta + (1-p) \left(\frac{x}{1+x} \right)^\beta}{1+x^\beta}$$

where $\beta = \frac{1}{\alpha}$.

Lemma 3.15. *When $p \leq \frac{1}{2}$ and $\alpha \in \mathbb{N}$, the supremum of $f(x, \beta, p)$ over x is attained at $x = 1$.*

Proof. Differentiate $f(x, \beta, p)$ to get

$$\begin{aligned}
\frac{\partial f(x, \beta, p)}{\partial x} &= \frac{\beta}{(1+x^\beta)^2} \left[(1+x^\beta) \left(p(1+x)^{\beta-1} + (1-p) \left(\frac{x}{1+x} \right)^{\beta-1} \left(\frac{1}{1+x} \right)^2 \right) \right. \\
&\quad \left. - x^{\beta-1} \left(p(1+x)^\beta + (1-p) \left(\frac{x}{1+x} \right)^\beta \right) \right] \\
&= \frac{\beta}{(1+x^\beta)^2} \left[p \left[(1+x)^{\beta-1}(1+x^\beta) - x^{\beta-1}(1+x)^\beta \right] + \right. \\
&\quad \left. (1-p) \left[\left(\frac{x}{1+x} \right)^{\beta-1} \left(\frac{1}{1+x} \right)^2 (1+x^\beta) - x^{\beta-1} \left(\frac{x}{1+x} \right)^\beta \right] \right] \\
&= \frac{\beta}{(1+x^\beta)^2} \left[p \left[(1+x)^{\beta-1} + (1+x)^{\beta-1} x^{\beta-1} (x - (1+x)) \right] + \right. \\
&\quad \left. (1-p) \left[\left(\frac{x^{\beta-1}}{(1+x)^{\beta+1}} \right) + \left(\frac{x^2}{1+x} \right)^{\beta-1} \left(\frac{x}{(1+x)^2} - \frac{1}{1+x} \right) \right] \right] \\
&= \frac{\beta}{(1+x^\beta)^2} \left[p(1+x)^{\beta-1}(1-x^{\beta-1}) + \right. \\
&\quad \left. (1-p) \left(\frac{x}{1+x} \right)^{\beta-1} \left(\frac{1}{1+x} \right)^2 (1-x^2) \right] \\
&= \frac{\beta}{(1+x^\beta)^2(1+x)^{\beta-1}} \left[p(1+x)^{2\beta}(1-x^{\beta-1}) + (1-p)x^{\beta-1}(1-x^2) \right].
\end{aligned}$$

For $x \leq 1$, $(1 - x^{\beta-1}) \leq 0$ (because $\beta \leq 1$) and similarly $(1 - x^2) \geq 0$. The opposite is the case for $x \geq 1$. So if we define

$$g(x, \beta) = (1 + x)^{2\beta}(1 - x^{\beta-1}) + x^{\beta-1}(1 - x^2)$$

then for $p \leq \frac{1}{2}$ we have for $0 < x \leq 1$

$$\frac{\partial f(x, \beta, p)}{\partial x} \geq \frac{\beta}{2(1 + x^\beta)^2(1 + x)^{\beta+1}} g(x, \beta)$$

and for $x \geq 1$ we have

$$\frac{\partial f(x, \beta, p)}{\partial x} \leq \frac{\beta}{2(1 + x^\beta)^2(1 + x)^{\beta+1}} g(x, \beta).$$

Hence the sign of $\frac{\partial f(x, \beta, p)}{\partial x}$ is the same as that of $g(x, \beta)$ for all $x > 0$. Now

$$g(x, \beta) = x^{\beta-1}[(1 - x^2) - (1 + x)^{2\beta}(1 - x^{1-\beta})]$$

and since $2 = (2\alpha)\beta$ and $1 - \beta = (\alpha - 1)\beta$ and 2α and $\alpha - 1$ are both integers, we can take a factor of $(1 - x^\beta)$ out as follows:

$$\begin{aligned} g(x, \beta) &= x^{\beta-1}(1 - x^\beta)[(1 + x^\beta + x^{2\beta} + \dots + x^{(2\alpha-1)\beta}) - (1 + x)^{2\beta}(1 + x^\beta + x^{2\beta} + \dots + x^{(\alpha-2)\beta})] \\ &= x^{\beta-1}(1 - x^\beta)[(1 + x^\beta + x^{2\beta} + \dots + x^{(\alpha-2)\beta})(1 + x - (1 + x)^{2\beta}) + x^{(\alpha-1)\beta} + x^{(2\alpha-1)\beta}] \end{aligned}$$

So when $\alpha \geq 2$ (and so $\beta \leq \frac{1}{2}$),

$$\begin{aligned} g(x, \beta) &> 0 \text{ when } 0 < x < 1 \\ &< 0 \text{ when } x > 1. \end{aligned}$$

So $f(x, \beta, p)$ is maximised over x by $f(1, \beta, p)$. We already have the result in the case $\alpha = 1$. \square

Corollary 3.16. *For all $p < \frac{1}{2}$, we have $X_1^{(n)} \rightarrow 0$, almost surely.*

Proof. Fix $\alpha \geq 2$. Use Lemma 3.15 to get that $\sqrt[\alpha]{X_1^{(n)}}$ converges almost surely to zero if $f(1, 1/\alpha, p) < \frac{1}{2}$. Now

$$2f(1, 1/\alpha, p) = p(\sqrt[\alpha]{2}) + (1 - p) \left(\frac{1}{\sqrt[\alpha]{2}} \right)$$

so we need

$$(2^{\frac{2}{\alpha}} - 1)p < 2^{1-\alpha} - 1$$

which gives

$$p < \frac{1}{2^{\frac{1}{\alpha}} + 1}.$$

But

$$\lim_{\alpha \rightarrow \infty} \frac{1}{2^{\frac{1}{\alpha}} + 1} = \frac{1}{2}$$

so the result follows. \square

Corollary 3.17. *For all $p > \frac{1}{2}$, we have $X_1^{(n)} \rightarrow \infty$, almost surely.*

Proof. This follows immediately from Lemma 3.6 and Corollary 3.16. \square

To get some more information about growth or decay rates, we take $\mu \in [\frac{1}{2}, 2]$ and consider the random hierarchical system generated by the functions

$$\begin{aligned} f_1^\mu(x_1, x_2) &= \mu^{-1}(x_1 + x_2) \\ f_2^\mu(x_1, x_2) &= \mu^{-1} \left(\frac{x_1 x_2}{x_1 + x_2} \right) \end{aligned}$$

which will give random variables $X_j^{(n,\mu)}$ with the same distribution (for the same value of p) as $\mu^{-n} X_j^{(n)}$ where $X_j^{(n)}$ is as in the original model (corresponding to $\mu = 1$). We now define $f(x, \beta, p)$ as for Lemma 3.15 and define $\epsilon(\alpha)$ such that

$$\frac{1}{2} + \epsilon(\alpha) = \sup_p \{f(1, \beta, p) \geq f(x, \beta, p) \text{ for all } x \geq 0\}$$

where $\beta = 1/\alpha$. Because f is continuous the supremum will be attained.

Lemma 3.18. *If $p < \frac{1}{2} + \epsilon(\alpha)$ and*

$$\mu > (2f(1, \frac{1}{\alpha}, p))^\alpha$$

for some $\alpha \in \mathbb{N}$ then

$$X_1^{(n,\mu)} \rightarrow 0$$

almost surely and furthermore this applies to any sequence of random variables with these distributions.

Proof. This follows from the fact that

$$\mathbb{E} \left(\sqrt[\alpha]{X_1^{(n,\mu)}} \right) \leq \left(2\mu^{-\frac{1}{\alpha}} f\left(1, \frac{1}{\alpha}, p\right) \right)^n$$

via the usual argument involving the Borel-Cantelli Lemma. \square

We note that Lemma 3.15 implies that $\epsilon(\alpha) \geq 0$ for $\alpha \in \mathbb{N}$. This is enough to give the following:

Corollary 3.19. *When $p = \frac{1}{2}$,*

$$\frac{1}{n} \log X_1^{(n)} \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.$$

Proof. Take $\mu > 1$. We require, for some $\alpha \in \mathbb{N}$,

$$\frac{1}{2} \left(\sqrt[\alpha]{2} + \sqrt[\alpha]{\frac{1}{2}} \right) < \mu^{\frac{1}{\alpha}}$$

but

$$\lim_{\alpha \rightarrow \infty} \left(\frac{1}{2} \left(\sqrt[\alpha]{2} + \sqrt[\alpha]{\frac{1}{2}} \right) \right)^\alpha = 1,$$

so for sufficiently large α we will have $\mu^{-n} X_j^{(n)} \rightarrow 0$ almost surely, from Lemma 3.18. We use the symmetry properties to obtain, for $\mu < 1$, $\mu^{-n} X_j^{(n)} \rightarrow \infty$ a.s., which gives the result. \square

Lemma 3.20. *We have that*

$$\epsilon(\alpha) > 0 \text{ for all } \alpha \in \mathbb{N}.$$

Proof. We know that $\epsilon(1) = \frac{1}{2}$. From the proof of Lemma 3.15 we see that the function $\hat{f}(x, \beta, p) = f(x, \beta, p) - f(1, \beta, p)$ is strictly negative when $p = \frac{1}{2}$, $\beta = 1/\alpha$, $\alpha \in \mathbb{N}$, except when $x = 1$. As this $\hat{f}(x, \beta, p)$ is infinitely differentiable in x and p (if $x > 0$), and $\hat{f}(1, \beta, p) = 0$ and $\frac{\partial \hat{f}}{\partial x} = 0$ for all β, p ,

to show that, for each $\alpha \in \mathbb{N}$, $\hat{f}(x, 1/\alpha, p) \leq 0$ for all x , for some $p > \frac{1}{2}$, it remains to show that

$$\frac{\partial^2 \hat{f}}{\partial^2 x}(1, \beta, \frac{1}{2}) < 0,$$

so that this second derivative will remain negative for p in a neighbourhood of $\frac{1}{2}$.

Now we can calculate

$$\frac{\partial^2 \hat{f}}{\partial^2 x}(1, \beta, \frac{1}{2}) = \frac{\beta}{2^{\beta+4}}(2^{2\beta}(1 - \beta) - \beta - 1).$$

We know $\frac{\partial^2 \hat{f}}{\partial^2 x}(1, \beta, \frac{1}{2}) \leq 0$, so we have the result as long as there are no solutions of

$$\beta = \frac{2^{2\beta} - 1}{2^{2\beta} + 1}$$

in $(0, 1)$. But

$$\frac{d}{d\beta} \left(\frac{2^{2\beta} - 1}{2^{2\beta} + 1} \right) = \frac{2^{2\beta+2} \log 2}{(2^{2\beta} + 1)^2}$$

and as $\frac{4y}{(y+1)^2} < 1$ for $y > 1$ the derivative is bounded above by $\log 2$ in $(0, 1)$, giving the result. \square

Corollary 3.21. *a) For $p > \frac{1}{2}$, there exists a value $\lambda(p) > 0$ such that, almost surely,*

$$(2p - 1) \log 2 < \frac{1}{n} \log X_1^{(n)} < \lambda(p) \text{ eventually.}$$

Furthermore, $\lambda(p) \rightarrow 0$ as $p \downarrow \frac{1}{2}$.

b) For $p < \frac{1}{2}$,

$$(2p - 1) \log 2 > \frac{1}{n} \log X_1^{(n)} > -\lambda(1 - p) \text{ eventually.}$$

Proof. Upper bound of part a):

We use Lemma 3.18 to see that, if $p < \frac{1}{2} + \epsilon(\alpha)$ and

$$\mu > \left(p \sqrt[\alpha]{2} + (1 - p) \sqrt[\alpha]{\frac{1}{2}} \right)^\alpha$$

then

$$\mu^{-n} X_1^{(n)} \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.$$

So we define

$$\lambda(p) = \inf_{p < \frac{1}{2} + \epsilon(\alpha)} \alpha \log \left(p \sqrt[\alpha]{2} + (1-p) \sqrt[\alpha]{\frac{1}{2}} \right).$$

Now we let $p \downarrow \frac{1}{2}$ and use Lemma 3.20 to allow us to take the limit as $\alpha \rightarrow \infty$.

The lower bound of part b) follows by symmetry.

Upper bound of part b):

For all $p < \frac{1}{2}$ and $\mu > \left(p \sqrt[\alpha]{2} + (1-p) \sqrt[\alpha]{\frac{1}{2}} \right)^\alpha$ we have $\mu^{-n} X_1^{(n)} \rightarrow 0$ a.s. as $n \rightarrow \infty$. Take the limit of μ as $\alpha \rightarrow \infty$, using l'Hôpital's rule, to get the result.

The lower bound of part a) follows by symmetry. \square

The proof of the following result uses a subadditivity technique:

Lemma 3.22. *We have that $\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} X_1^{(n)}$ exists.*

Proof. We assume $p \geq \frac{1}{2}$.

We define the σ -algebras

$$\mathcal{G}_{n,n+m} = \sigma(Z_k^{(j)}, k \in \mathbb{N}, n < j \leq n+m) \text{ for } m > 0.$$

Also for $m \geq 0$, we now define the random variables $R_k^{(n,n+m)}$ as follows:

$$R_k^{(n,n)} = 1 \text{ for all } n, k.$$

$$R_k^{(n,n+m+1)} = f_{Z_k^{(n+m+1)}}(R_{2k-1}^{(n,n+m)}, R_{2k}^{(n,n+m)}) \text{ for } m \geq 0, k \in \mathbb{N}.$$

Intuitively $R_k^{(n,n+m)}$ represents the last m stages of the construction of the network. If the initial random variables $X_k^{(0)}$ are a.s. equal to 1 then $R_k^{(n,n+m)}$ has the same distribution as $X_k^{(m)}$.

Now

$$X_1^{(n+m)} = a(X_1^{(n)}, \dots, X_{2^m}^{(n)})$$

where a is a $\mathcal{G}_{n,n+m}$ -measurable random concave function of 2^m variables, such that $a(1, 1, \dots, 1) = R_1^{(n,n+m)}$.

Now

$$\mathbb{P}(X_1^{(n)} = x_1, \dots, X_{2^m}^{(n)} = x_{2^m}) = \mathbb{P}(X_{\sigma(1)}^{(n)} = x_1, \dots, X_{\sigma(2^m)}^{(n)} = x_{2^m}) \text{ for all } \sigma \in S_{2^m}$$

so if we take the 2^m random variables $\{X_1^{(n)}, \dots, X_{2^m}^{(n)}\}$ in increasing order as $X_{2^m:1}^{(n)}, X_{2^m:2}^{(n)}, \dots, X_{2^m:2^m}^{(n)}$ and define $\mathcal{H}_{n,m} = \sigma(X_{2^m:1}^{(n)}, X_{2^m:2}^{(n)}, \dots, X_{2^m:2^m}^{(n)})$ then we have

$$\mathbb{E}(X_1^{(n+m)} | \mathcal{G}_{n,m}, \mathcal{H}_{n,m}) = \hat{a}(X_{2^m:1}^{(n)}, X_{2^m:2}^{(n)}, \dots, X_{2^m:2^m}^{(n)})$$

where \hat{a} is the symmetric concave homogeneous function

$$\hat{a}(x_1, \dots, x_{2^m}) = \frac{1}{(2^m)!} \sum_{\sigma \in S_{2^m}} a \cdot \sigma(x_1, \dots, x_{2^m})$$

which will be bounded above by its linear approximation at $(1, 1, \dots, 1)$ so

$$\hat{a}(x_1, \dots, x_{2^m}) \leq \frac{R_1^{(n,n+m)}}{2^m} \sum_{k=1}^{2^m} X_k^{(n)}.$$

Hence we can conclude that

$$\mathbb{E}X_1^{(n+m)} \leq \mathbb{E}R_1^{(n,n+m)} \mathbb{E}X_1^{(n)} = \mathbb{E}X_1^{(m)} \mathbb{E}X_1^{(n)}$$

and so conclude the result, by the subadditivity of the sequence $(\log \mathbb{E}X_n^{(1)})_{n \in \mathbb{N}}$. For $p < \frac{1}{2}$ the result follows by self-duality. \square

3.2.4 Another related hierarchical system

Another variation on the series-parallel network is to modify the functions as follows:

$$\begin{aligned} f_1(x_1, x_2) &= \mu(x_1 + x_2) \\ f_2(x_1, x_2) &= \frac{x_1 x_2}{\mu(x_1 + x_2)}. \end{aligned}$$

The case $\mu = 1$ corresponds to the hierarchical system associated with the resistance of the series-parallel network already investigated. In general we can consider this as modelling the resistance of a sequence of weighted graphs obtained by weighting the resistance of graphs connected in series by μ and of those connected in parallel by $1/\mu$.

We note that the case $\mu = \frac{1}{2}$ has point masses as fixed point distributions and that convergence to these will occur by the results of Chapter 2.

The arguments for convergence to 0 and ∞ when $\mu = 1$ extend to $\mu > \frac{1}{2}$ generally.

The variance results (Theorem 3.10) work out somewhat differently. When $p = \frac{1}{2}$ the variance still satisfies

$$\text{Var } Y_1^{(n)} = \frac{1}{2} \text{Var } Y_1^{(n-1)} + \text{Var} \left(Y_1^{(n)} - \frac{Y_1^{(n-1)} + Y_2^{(n-1)}}{2} \right)$$

and

$$\text{Var} \left(Y_1^{(n)} - \frac{Y_1^{(n-1)} + Y_2^{(n-1)}}{2} \right) = \mathbb{E} \left(\log \frac{X_1^{(n)}}{\sqrt{X_1^{(n-1)} X_2^{(n-1)}}} \right)^2.$$

So the increment is

$$\begin{aligned} & \int \frac{1}{2} \left(\left(\log \frac{\mu(x_1 + x_2)}{\sqrt{x_1 x_2}} \right)^2 + \left(\log \frac{\sqrt{x_1 x_2}}{\mu(x_1 + x_2)} \right)^2 \right) d\nu_{n-1}(x_1) d\nu_{n-1}(x_2) \\ &= \int \left(\log \frac{\mu(x_1 + x_2)}{\sqrt{x_1 x_2}} \right)^2 d\nu_{n-1}(x_1) d\nu_{n-1}(x_2) \\ &= (\log \mu)^2 + 2 \int \log \mu \log \left(\frac{x_1 + x_2}{\sqrt{x_1 x_2}} \right) d\nu_{n-1}(x_1) d\nu_{n-1}(x_2) \\ & \quad + \int \left(\log \frac{(x_1 + x_2)}{\sqrt{x_1 x_2}} \right)^2 d\nu_{n-1}(x_1) d\nu_{n-1}(x_2). \end{aligned}$$

The last term here was analysed in the proof of Theorem 3.10, so assuming the log-symmetry of the distribution we can conclude that

$$\begin{aligned} \text{Var } Y_1^{(n)} - \text{Var } Y_1^{(n-1)} &= (\log \mu)^2 + 2(\log \mu) \mathbb{E} \log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{\sqrt{X_1^{(n-1)} X_2^{(n-1)}}} \right) \\ &\quad + \mathbb{E} \left(\log(X_1^{(n-1)} + X_2^{(n-1)}) \log \left(\frac{X_1^{(n-1)} + X_2^{(n-1)}}{X_1^{(n-1)} X_2^{(n-1)}} \right) \right). \end{aligned}$$

We know from the proof of Theorem 3.10 that the last term here is positive. However the second term is negative (for $\mu < 1$). As $\text{Var } Y_1^{(n)} \rightarrow 0$ as $n \rightarrow \infty$ when $\mu = \frac{1}{2}$ this expression must be negative for n sufficiently large. As, for a given distribution, the expression is continuous in μ this shows that it can also be negative for some distributions and some $\mu \in (\frac{1}{2}, 1)$. This suggests that in this case we may not have $\text{Var } Y_1^{(n)} \rightarrow \infty$.

3.3 More examples of random fractal graphs

3.3.1 Introduction

We now consider some more complicated examples of random fractal graphs where the model graphs Γ_i have three edges.

If a hierarchical system $\{X_j^{(n)}\}$ is converted into another system $\{Y_j^{(n)}\}$ by the substitution $Y_j^{(n)} = (X_j^{(n)})^{-1}$ then we will consider the systems to be *dual*. For example the resistance of the series-parallel network is self-dual, which is useful in analysing it.

As before, let $G^{(n)}$ be the graph at stage n in the replacement construction and let $R^{(n)}$ be the resistance in $G^{(n)}$ between vertices 1 and 2.

For all the networks below, we assume that $X_j^{(0)} = 1$ for all j and that the other random variables are defined iteratively by a random hierarchical system with probability p on f_1 and probability $1 - p$ on f_2 .

3.3.2 A non-self-dual network

The model graphs Γ_1 and Γ_2 are those in Figure 3.3 with probabilities p and $1 - p$ respectively.



Figure 3.3: The model graphs for network 3.3.2

The functions for the hierarchical system are now

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \\ f_2(x_1, x_2, x_3) &= \frac{x_1(x_2 + x_3)}{x_1 + x_2 + x_3} \end{aligned}$$

As for Lemma 3.3, we note that if $q = \mathbb{P}(X_1^{(n)} \rightarrow \infty)$ then q satisfies

$$\begin{aligned} q &= p(1 - (1 - q)^3) + (1 - p)(q(1 - (1 - q)^2)) \\ &= p(3q - 3q^2 + q^3) + (1 - p)(2q^2 - q^3) \end{aligned}$$

so that the polynomial

$$h(q) = q^3(p - (1 - p)) + q^2(2(1 - p) - 3p) + q(3p).$$

If $h(q) = q$ then

$$q^3(p - (1 - p)) + q^2(2(1 - p) - 3p) + q(3p - 1) = 0$$

and so either $q = 0$ or

$$q^2(2q - 1) + q(2 - 5p) + (3p - 1) = 0.$$

If $p = \frac{1}{2}$ then q must be 0 or 1; otherwise

$$q = \frac{5p - 2 \pm \sqrt{4 + 25p^2 - 24p^2 - 4 + 20p - 20p}}{4p - 2} = \frac{5p - 2 \pm p}{4p - 2}$$

which is 1 or $\frac{3p-1}{2p-1}$. So if $p \geq \frac{1}{3}$ then q follows a zero-one law; if $p < \frac{1}{3}$ then the value $\frac{3p-1}{2p-1}$ is also a possibility. Note however that this is a strictly decreasing function of p in this range, so it cannot be the probability of convergence to ∞ for a range of p , as the latter is an increasing function of p .

The branching process $C^{(n)}$, defined equivalently to in Definition 3.1 has an offspring mean of $3p$, so is supercritical for $p > \frac{1}{3}$. This is enough to show that $R^{(n)} \rightarrow \infty$ almost surely for $p > \frac{1}{3}$. However the other branching process $\tilde{C}^{(n)}$ is never supercritical (it has offspring mean $1 - p$).

We now look at the behaviour of the hierarchical system, which gives the following result.

Proposition 3.23. *For $p < \frac{1}{7}$, both $X_1^{(n)} \rightarrow 0$ and $R^{(n)} \rightarrow 0$, almost surely.*

Proof. To symmetrise the system, we replace f_2 with a probability $\frac{1-p}{3}$ on each of the functions

$$\begin{aligned} f_{2,1}(x_1, x_2, x_3) &= \frac{x_1(x_2 + x_3)}{x_1 + x_2 + x_3} \\ f_{2,2}(x_1, x_2, x_3) &= \frac{x_2(x_1 + x_3)}{x_1 + x_2 + x_3} \\ f_{2,3}(x_1, x_2, x_3) &= \frac{x_3(x_2 + x_1)}{x_1 + x_2 + x_3} \end{aligned}$$

and use this to construct a set of random variables $(\hat{X}_j^{(n)})$. Note that

$$\frac{1}{3}(f_{2,1}(x_1, x_2, x_3) + f_{2,2}(x_1, x_2, x_3) + f_{2,3}(x_1, x_2, x_3)) \leq \frac{2}{9}(x_1 + x_2 + x_3)$$

so

$$\mathbb{E}\hat{X}_1^{(n)} \leq (3p + \frac{2}{3}(1-p))\mathbb{E}\hat{X}_1^{(n-1)} = \left(\frac{7}{3}p + \frac{2}{3}\right)\mathbb{E}\hat{X}_1^{(n-1)}$$

so the growth/decay rate in the mean is at most $\frac{7p+2}{3}$. Hence if $p < \frac{1}{7}$ there is exponential decay of the mean and hence almost sure convergence of $\hat{X}_1^{(n)}$ to 0.

The exponential decay of the expectations allows us to conclude that this a.s. convergence applies to any set of random variables with these distributions, so that the original random variables $X_1^{(n)} \rightarrow 0$, and also that $R^{(n)} \rightarrow 0$, almost surely. \square

Note that this technique can be used in general (for concave functions) to show that the growth/decay rate is bounded above by $\mathbb{E}\lambda_{Z_j^{(n)}}$ where $\lambda_j = f_j(1, 1, 1)$ and hence to find a region of a.s. convergence to zero.

Here $\lambda_1 = 3$ and $\lambda_2 = \frac{2}{3}$.

We now consider the dual to this network. This uses the following functions:

$$f_1(x_1, x_2, x_3) = \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \right)^{-1}$$

$$f_2(x_1, x_2, x_3) = x_1 + \frac{x_2 x_3}{x_2 + x_3}$$

which correspond to the network elements in Figure 3.4. Here $\lambda_1 = \frac{1}{3}$ and



Figure 3.4: The model graphs for the dual to network 3.3.2

$\lambda_2 = \frac{3}{2}$.

We calculate

$$\mathbb{E}\lambda_{Z_j^{(n)}} = \frac{1}{3}p + \frac{3}{2}(1-p) = \frac{9-7p}{6},$$

so we have a.s. convergence to 0 when $p > \frac{3}{7}$.

Hence we know that, for network 3.3.2, we have the following results

- When $p < \frac{1}{7}$, both $R^{(n)}$ and $X_1^{(n)}$ converge almost surely to 0.
- When $p \in (\frac{1}{3}, \frac{3}{7}]$, $R^{(n)} \rightarrow \infty$ almost surely. This gives a distributional convergence for $X_1^{(n)}$ but not necessarily almost sure convergence.
- When $p > \frac{3}{7}$, both $R^{(n)}$ and $X_1^{(n)}$ converge almost surely to ∞ .
- When $p \in [\frac{1}{7}, \frac{1}{3}]$ the situation is unclear.

3.3.3 More rapid growth or decay

The model graphs Γ_1 and Γ_2 are those in Figure 3.5 with probabilities p and $1 - p$ respectively.



Figure 3.5: The model graphs for network 3.3.3

The functions for the hierarchical system are now

$$f_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

$$f_2(x_1, x_2, x_3) = \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \right)^{-1}$$

Here $\lambda_1 = 3$ and $\lambda_2 = \frac{1}{3}$. This model is self-dual, so results about its behaviour with small p can be used to deduce results about its behaviour with large p and vice versa.

The polynomial

$$h(q) = p(3q - 3q^2 + q^3) + (1 - p)q^3$$

so setting $h(q) = q$ gives either $q = 0$ or $q^2 - 3pq^2 + (3p - 1)q = 0$, with roots $q = 0, 1$ or $3p - 1$. This last root is only in the range $[0, 1]$ if $p \in [\frac{1}{3}, \frac{2}{3}]$.

The branching process $C^{(n)}$, again defined analogously to Definition 3.1 is supercritical for $p > \frac{1}{3}$ as for network 3.3.2. Furthermore the other branching process $\tilde{C}^{(n)}$ is supercritical for $p < \frac{2}{3}$ (Note that these two ranges overlap).

This allows us to conclude that, for $p \leq \frac{1}{3}$, $R^{(n)} \rightarrow 0$ and, for $p \geq \frac{2}{3}$, $R^{(n)} \rightarrow \infty$, almost surely. Furthermore for $p \in (\frac{1}{3}, \frac{2}{3})$ we have that $R^{(n)} \rightarrow \infty$ with probability $3p - 1$ and that $C^{(n)} \rightarrow 0$ with probability $2 - 3p$.

Using arguments similar to Proposition 3.23 we see that, for $p < \frac{1}{4}$, $\mathbb{E}X_1^{(n)}$ decays exponentially (because $\mathbb{E}\lambda_{Z_1^{(n)}} = 3p + \frac{1}{3}(1 - p) = \frac{8p+1}{3}$). Hence $X_1^{(n)} \rightarrow 0$ almost surely for $p < \frac{1}{4}$ and by duality $X_1^{(n)} \rightarrow \infty$ almost surely for $p > \frac{3}{4}$. In the intermediate range we only have a convergence in distribution result (to a degenerate law).

3.3.4 Another self-dual example

The model graphs Γ_1 and Γ_2 are those in Figure 3.6 with probabilities p and $1 - p$ respectively.

The functions for the hierarchical system are now

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1 + \frac{x_2 x_3}{x_2 + x_3} \\ f_2(x_1, x_2, x_3) &= \frac{x_1(x_2 + x_3)}{x_1 + x_2 + x_3} \end{aligned}$$



Figure 3.6: The model graphs for network 3.3.4

Here $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = \frac{2}{3}$. Like model 3.3.3, this one is self-dual.

For this network, the branching processes $C^{(n)}$ and $\tilde{C}^{(n)}$ both have offspring distribution concentrated on $\{0, 1\}$ so are never supercritical.

The polynomial $h(q)$ here is

$$q^3 + (p - 2)q^2 + pq,$$

with the roots of $h(q) = q$ being 0, 1 and $1 - p$.

We can calculate

$$\mathbb{E}\lambda_{Z_j^{(n)}} = \frac{3}{2}p + \frac{2}{3}(1 - p) = \frac{5p + 4}{6},$$

so almost sure convergence of $X_1^{(n)}$ to 0 occurs when $p < \frac{2}{5}$. Self duality implies that almost sure convergence to ∞ occurs when $p > \frac{3}{5}$.

The methods used here leave several possibilities open for the range $[\frac{2}{5}, \frac{3}{5}]$ including a non-degenerate limit either at $p = \frac{1}{2}$ or in some larger interval, but also degenerate limits with mass at 0 and ∞ or even some more complex behaviour.

Chapter 4

The length of the network

4.1 Introduction

Resistance is not the only property of random fractal graphs which can be studied using random hierarchical systems. Another example is the graph distance between the two boundary points (in a two point graph). Turning specifically to the non-homogeneous series-parallel network and the replacement construction, we label the vertices as follows: the initial vertices are 1 and 2, and new vertices are numbered in order of their insertion i.e. when an edge $a \leftrightarrow b$ is replaced by two edges in series these are $a \leftrightarrow c$ and $c \leftrightarrow b$ with $c > a, b$. We will define $j(c)$ to be the stage at which vertex c was added to the graph i.e. $j(c) = \min_n \{c \in V(G^{(n)})\}$ and further define $n(c)$ to be a neighbour of c in $G^{(j(c))}$ i.e. $n(c) = \min_a \{a \leftrightarrow c \text{ in } G^{(j(c))}\}$. We take $n(1) = 2$ and $n(2) = 1$, and $j(1) = j(2) = 0$.

4.2 The map κ_p

Consider a replacement construction $\{G^{(n)}\}_{n \in \mathbb{N}}$ of a random fractal graph. We define the distance $d^{(n)}(a, b)$ to be the graph distance in $G^{(n)}$ between a

and b , for $a, b \in V(G^{(n)})$.

For a random fractal graph with just two boundary points, we now define $K^{(n)} = d^{(n)}(1, 2)$, the distance between the two boundary points. This can be thought of as the ‘length’ of the network.

We now consider the behaviour of this length for the series-parallel network.

Lemma 4.1. *a) If $p > \frac{1}{2}$ then $K^{(n)}$ grows exponentially quickly with probability 1.*

b) If $p < \frac{1}{2}$ then $K^{(n)}$ has a finite limit K as $n \rightarrow \infty$, with probability 1.

Proof. Recall the branching processes defined in Definition 3.1.

For a), we use the branching process $C^{(n)}$ from Definition 3.1 to show that the probability is positive, because $K^{(n)} \geq C^{(n)}$, and then use Lemma 3.3 on the event $K^{(n)} \rightarrow \infty$ to see that it must be 1.

For b), we use the other branching process $\tilde{C}^{(n)}$ which is the number of edges $1 \leftrightarrow 2$. This is supercritical when $p < \frac{1}{2}$, so with positive probability we have $K^{(n)} = 1$ for all n . Now we use Lemma 3.3 again on the same event to get the result. \square

Corollary 4.2. *When $p < \frac{1}{2}$, the hierarchical system with functions*

$$\begin{aligned} f_1(x_1, x_2) &= x_1 + x_2 \\ f_2(x_1, x_2) &= \min(x_1, x_2), \end{aligned}$$

with probabilities p and $1 - p$ respectively, has a non-degenerate fixed point.

Proof. The law of the random variable K gives such a fixed point. \square

Note that this gives a non-degenerate fixed point whenever we have a network where the branching process $\tilde{C}^{(n)}$ is supercritical and that there are no roots of $h(q) = q$ in the interval $(0, 1)$, where $h(q)$ is the polynomial discussed after the proof of Lemma 3.3.

We will define κ_p to be the map on probability measures on \mathbb{R}^+ defined by the hierarchical system in Corollary 4.2, so that if the random variables $K^{(0)}$ have law $\nu^{(0)}$, the random variables $K^{(n)}$ have law $\kappa_p^n(\nu^{(0)})$.

We will now investigate the properties of K further.

Lemma 4.3. *For $p < \frac{1}{2}$, if the law of K is a fixed point on \mathbb{R}^+ of κ_p then K has tail probabilities $\mathbb{P}(K > x) = o(x^{-s})$ for $s < -(\frac{\log p}{\log 2} + 1)$.*

Proof. Let L be an independent copy of K . Then

$$\mathbb{P}(K > x) = p\mathbb{P}(K + L > x) + (1 - p)\mathbb{P}(K > x)^2.$$

For $p < p_0$, we have $p(1 + \epsilon) < p_0(1 - \epsilon)$ for some $\epsilon > 0$. We will be able to take x large enough so that $(1 - (1 - p)\mathbb{P}(K > x))^{-1} < 1 + \epsilon$.

Now set

$$\begin{aligned} r_1(x) &= \frac{\mathbb{P}(K > x)}{\mathbb{P}(K > \frac{x}{2})} \\ &= p \frac{\mathbb{P}(K + L > x)}{\mathbb{P}(K > \frac{x}{2})} + (1 - p) \frac{\mathbb{P}(K > x)^2}{\mathbb{P}(K > \frac{x}{2})} \\ &\leq 2p + (1 - p)\mathbb{P}(K > x)r_1(x) \end{aligned}$$

using the inequality

$$\mathbb{P}(K + L > x) \leq 2\mathbb{P}(K > \frac{x}{2}). \quad (4.1)$$

This gives $2p \geq r_1(x)(1 - (1 - p)\mathbb{P}(K > x))$ and hence

$$r_1(x) \leq \frac{2p}{1 - \epsilon} < \frac{2p_0}{1 - \epsilon}$$

for large enough x . So, when $p < p_0$, we have that $\mathbb{P}(K > x) = o(x^{(\frac{\log p_0}{\log 2} + 1)})$. This gives the result as stated. \square

Corollary 4.4. *If the law of K is a fixed point (on \mathbb{R}^+) of κ_p then K has finite mean if $p < \frac{1}{4}$.*

To improve this result to $p < \frac{1}{2}$ we would need to improve the estimate (4.1).

4.3 The fixed points of κ_p

We now consider the iteration of κ_p . For $p < \frac{1}{2}$, one fixed point is the distribution ν_p of the random variable K found in Lemma 4.1. We will see that for this range of p this fixed point is essentially unique while for $p > \frac{1}{2}$ there are no non-degenerate fixed points for distributions on \mathbb{R}^+ .

We consider labelling the edges of the graph $G^{(n)}$ with i.i.d. random variables on \mathbb{R}^+ with distribution $\mu^{(0)}$. We let $L^{(n)}$ be the minimum sum of the labels on a route between the endpoint vertices. This describes first-passage percolation on the graph $G^{(n)}$.

Also let $\mu^{(n)} = \kappa_p^n(\mu^{(0)})$.

Lemma 4.5. *The distribution of $L^{(n)}$ is $\mu^{(n)}$.*

Proof. The proof is similar to that of Lemma 1.5. The statement is obvious for $n = 0$. For larger n we assume the statement for $n - 1$ and note that $G^{(n)}$ consists of two i.i.d. graphs with the distribution of $G^{(n-1)}$ connected in series with probability p and in parallel with probability $1 - p$. In the former case $L^{(n)}$ is the sum of two independent variables with distribution $\mu^{(n-1)}$ and in the latter case it is the minimum of these variables. This describes the map κ_p so proves the lemma by induction. \square

We now investigate the sequence $(L^{(n)})_{n \in \mathbb{N}}$. We define λ to be the infimum of the support of $\mu^{(0)}$ i.e. $\lambda = \inf\{x : \mu^{(0)}(-\infty, x) > 0\}$. Note that the law $\mathcal{L}(\lambda K)$ of λK is a fixed point of κ_p for $p < \frac{1}{2}$.

Theorem 4.6. *For $p < \frac{1}{2}$, as $n \rightarrow \infty$,*

$$\kappa_p^n(\mu^{(n)}) \rightarrow \mathcal{L}(\lambda K) \text{ (weak convergence).}$$

Proof. We condition on the (replacement) sequence of graphs $G^{(n)}$. We know from Lemma 4.1 that for n large enough the ‘length’ $K^{(n)} = K$. First consider the case where $K = 1$. In this case the branching process $\tilde{C}^{(n)}$ from Definition 3.1 grows exponentially almost surely. The value $L^{(n)}$ is bounded above by

the minimum label on these $\tilde{C}^{(n)}$ edges. But this minimum value converges weakly to λ .

Similarly when $K = k$, we consider n_0 large enough that $d^{(n_0)}(1, 2) = k$. We consider the paths of length k between vertices 1 and 2 in $G^{(n_0)}$, and note that $K = k$ implies that one such path must be preserved in $G^{(n)}$ for $n > n_0$. Now the self-similarity of the structure shows that each edge in this path has a branching process associated with it, with the same offspring distribution as $\tilde{C}^{(n)}$, and also growing exponentially conditioned on survival. Hence the number of edge-disjoint paths from 1 to 2 of length k grows exponentially, and hence the weak convergence result holds.

This shows that $\mathbb{P}(L^{(n)} > \lambda K) \rightarrow 0$ as $n \rightarrow \infty$. Finally it is obvious that $\mathbb{P}(L^{(n)} < \lambda K^{(n)}) = 0$, completing the proof. \square

This shows that these are the only fixed points and that starting with any distribution on \mathbb{R}^+ the sequence will converge to one of them.

Note that if $\lambda = 0$ the limit is a point mass at 0.

For $p \geq \frac{1}{2}$ the lower bound in the proof shows that if $\lambda > 0$ then $L^{(n)} \rightarrow \infty$ because in that case $K^{(n)} \rightarrow \infty$. Furthermore if $p > \frac{1}{2}$ then $K^{(n)}$ grows exponentially fast almost surely, which is enough to show that the only fixed point is a point mass at 0.

In the $p < \frac{1}{2}$ case this shows that in the limit first-passage percolation on the series-parallel network loses all randomness except that coming from the random structure of the network.

Chapter 5

Another system with a non-degenerate fixed point

5.1 Introduction

The following hierarchical system is in some ways more tractable than the others we have studied because of the relatively simple functional equation in Lemma 5.1. We consider the random hierarchical system generated by the functions

$$f_1(x_1, x_2) = 2 \min(x_1, x_2)$$

and

$$f_2(x_1, x_2) = \frac{1}{2} \max(x_1, x_2).$$

We will consider the (log-symmetric) case with equal probability on each function.

The above defines a function γ on the space of distribution functions. We will investigate the possibility of the existence of a fixed-point distribution for γ . Let F be a distribution function for such a fixed point.

5.2 A recurrence

Lemma 5.1. *The distribution function F of a fixed point satisfies*

$$F(2a) = \sqrt{\left(F\left(\frac{a}{2}\right)\right)^2 + 2F(a) - 2F\left(\frac{a}{2}\right)}.$$

Proof. For $a \in \mathbb{R}$,

$$\begin{aligned} F(a) &= \mathbb{P}(X_1^{(1)} \leq a) \\ &= \frac{1}{2} \left(\mathbb{P}(\{X_1^{(0)} \leq \frac{a}{2}\} \cup \{X_1^{(0)} \leq \frac{a}{2}\}) + \mathbb{P}(\{X_1^{(0)} \leq 2a\} \cap \{X_1^{(0)} \leq 2a\}) \right) \\ &= \frac{1}{2} \left(1 - \left(1 - F\left(\frac{a}{2}\right)\right)^2 + (F(2a))^2 \right). \\ &= F\left(\frac{a}{2}\right) + \frac{1}{2} \left(F(2a)^2 - F\left(\frac{a}{2}\right)^2 \right). \end{aligned}$$

Rearranging gives the result. □

If we set $x_n = F(2^n)$ for $n \in \mathbb{Z}$ then we now have the recurrence

$$x_{n+2} = \sqrt{x_n^2 + 2(x_{n+1} - x_n)}. \tag{5.1}$$

We note that if we set $y_n = 1 - x_{-n}$ then y_n satisfies the same recurrence, and that, for any q ,

$$\sqrt{(1-q)^2 + 2\left(\frac{1}{2} - (1-q)\right)} = \sqrt{q^2 - 2q + 1 + 1 - 2 + 2q} = q,$$

so to find a solution to the recurrence it will suffice to find a solution for positive n with initial conditions $x_0 = \frac{1}{2}$, $x_1 = q$ and then to take $x_{-n} = 1 - x_n$. This can give a probability distribution only if x_n is increasing in n and $x_n \in [0, 1]$ for all n .

We assume that $x_m \geq x_{m-1}$ and $x_m \leq 1$ for $m \leq n + 1$. This implies that $x_m \geq q$ for $1 \leq m \leq n + 1$.

Lemma 5.2. *If $q \in (\frac{1}{2}, \frac{2}{3}]$ there exists a solution which gives a distribution function.*

Proof. Note that, by completing the square, (5.1) implies that

$$x_{m+2} \leq x_m + \left(\frac{x_{m+1}}{x_m} - 1 \right)$$

so that

$$x_{m+2} - x_{m+1} \leq (x_{m+1} - x_m) \left(\frac{1}{x_m} - 1 \right).$$

Hence for $m \leq n+1$

$$\begin{aligned} (x_{m+2} - x_{m+1}) &\leq (x_{m+1} - x_m)(1/q - 1) \\ &\leq (x_2 - x_1)(1/q - 1)^{m-1} \\ &\leq \left(q - \frac{1}{2} \right) (1/q - 1)^{m-1} \\ x_{m+2} &\leq \frac{2q-1}{2} \left(1 + \sum_{k=0}^{m-1} \left(\frac{1}{q} - 1 \right)^k \right) + \frac{1}{2} \\ &\leq \frac{2q-1}{2} \left(1 + \left(1 - \frac{1-q}{q} \right)^{-1} \right) + \frac{1}{2} \\ &= \left(\frac{2q-1}{2} \right) \left(\frac{3q-1}{2q-1} \right) + \frac{1}{2} \\ &= \frac{3q}{2}. \end{aligned}$$

So if $q \in (\frac{1}{2}, \frac{2}{3}]$ then for $m < n+3$ we have $x_m \leq 1$. We now rearrange (5.1) to obtain

$$x_{n+2} - x_n = \frac{2(x_{n+1} - x_n)}{x_{n+2} + x_n}$$

so

$$x_{n+2} - x_{n+1} = (x_{n+1} - x_n) \left(\frac{2}{x_{n+2} + x_n} - 1 \right).$$

But we know that $x_{n+2} + x_n \leq 2$, and by our induction hypothesis $x_{n+1} - x_n \geq 0$, so we see that $x_{n+2} - x_{n+1} \geq 0$.

Hence we have an increasing sequence $(x_n)_{n \in \mathbb{Z}}$ satisfying the recurrence with $x_n \in [0, 1]$ for all n . \square

We now obtain a probability distribution function on \mathbb{R}^+ which gives a fixed point by setting $F(x) = x_n$ for all $x \in [\frac{1}{\sqrt{2}}2^n, \sqrt{2}2^n)$.

Note that if $\lim_{x \rightarrow \infty} F(x) < 1$ then we can obtain a new log-symmetric fixed point by conditioning on the random variable being finite and non-zero.

Numerical calculations suggest that for the finite non-zero case $x_1 = F(2) = \sqrt{3}/2$, but we have no proof.

The fixed point constructed by this argument is concentrated on points of the form $2^m \sqrt{2}$ (with $m \in \mathbb{Z}$) but the argument does not rule out other fixed points.

5.3 Use of a metric

Consider the metric d on distribution functions defined by

$$d(F, G) = \int_0^\infty |F(e^x) - G(e^x)| dx.$$

Lemma 5.3. *The map γ is non-expansive (and hence continuous) in d , i.e.*

$$d(\gamma F, \gamma G) \leq d(F, G).$$

Furthermore equality holds only when

$$(F(2a) - G(2a)) \left(F\left(\frac{a}{2}\right) - G\left(\frac{a}{2}\right) \right) \geq 0 \text{ for all } a \in \mathbb{R}^+. \quad (5.2)$$

Proof. We see that

$$\begin{aligned} |\gamma F(a) - \gamma G(a)| &= \left| \frac{1}{2} \left(F(2a)^2 + 2F\left(\frac{a}{2}\right) - F\left(\frac{a}{2}\right)^2 - (F(2a)^2 - 2G\left(\frac{a}{2}\right) + G\left(\frac{a}{2}\right)^2) \right) \right| \\ &= \left| \frac{1}{2} \left((F(2a) - G(2a))(F(2a) + G(2a)) + \left(F\left(\frac{a}{2}\right) - G\left(\frac{a}{2}\right) \right) \left(2 - F\left(\frac{a}{2}\right) - G\left(\frac{a}{2}\right) \right) \right) \right| \\ &\leq \frac{1}{2} |F(2a) - G(2a)| (F(2a) + G(2a)) + \frac{1}{2} \left| F\left(\frac{a}{2}\right) - G\left(\frac{a}{2}\right) \right| \left(2 - F\left(\frac{a}{2}\right) - G\left(\frac{a}{2}\right) \right) \end{aligned}$$

with equality if and only if

$$(F(2a) - G(2a)) \left(F\left(\frac{a}{2}\right) - G\left(\frac{a}{2}\right) \right) \geq 0.$$

So,

$$\begin{aligned} d(\gamma F, \gamma G) &\leq \int_0^\infty \frac{1}{2} |F(2e^x) - G(2e^x)| (F(2e^x) + G(2e^x)) dx + \\ &\quad \int_0^\infty \frac{1}{2} \left| F\left(\frac{e^x}{2}\right) - G\left(\frac{e^x}{2}\right) \right| \left(2 - F\left(\frac{e^x}{2}\right) - G\left(\frac{e^x}{2}\right) \right) dx. \end{aligned}$$

Now a change of variables to $x - \log 2$ in the first integral and to $x + \log 2$ in the second integral gives

$$d(\gamma F, \gamma G) \leq d(F, G)$$

with equality if and only if (5.2) holds. □

So a set of distributions which is closed under γ and compact in the metric d will contain a fixed point of γ using the Brouwer fixed point theorem. Furthermore such a set is provided by those distribution functions F satisfying $d(F, F_0) < C$ for any fixed point F_0 and a constant C . The fixed point found by the recurrence method provides a known example for F_0 .

5.4 Properties preserved by γ

Lemma 5.4. *The map γ preserves $\mathbb{E} \log X$, i.e. $\mathbb{E} \log X_{\gamma F} = \mathbb{E} \log X_F$.*

Proof. If $X_1^{(0)}$ and $X_2^{(0)}$ are i.i.d. such that $\mathbb{E} \log X_1^{(0)}$ exists then

$$\begin{aligned}
\mathbb{E} \log X_1^{(1)} &= \frac{1}{2} \left(\mathbb{E} \log 2 \min(X_1^{(0)}, X_2^{(0)}) + \mathbb{E} \log \frac{1}{2} \max(X_1^{(0)}, X_2^{(0)}) \right) \\
&= \mathbb{E} \left(\frac{1}{2} (\log \min(X_1^{(0)}, X_2^{(0)}) + \log \max(X_1^{(0)}, X_2^{(0)})) + \log 2 + \log \frac{1}{2} \right) \\
&= \mathbb{E} \left(\frac{1}{2} (\log X_1^{(0)} + \log X_2^{(0)}) \right) \\
&= \mathbb{E} \log X_1^{(0)}
\end{aligned}$$

□

Now let $\phi(x) = \log_2 x - \lfloor \log_2 x \rfloor$ (where $\lfloor a \rfloor$ is the integer part of a) so that \mathbb{R}^+ is mapped to a circle by ϕ . We find that

Lemma 5.5. *The distribution of $\phi(X)$ is preserved by γ .*

Proof.

$$\begin{aligned}
\mathbb{P}(\phi(X_1^{(1)}) \leq a) &= \frac{1}{2} \left(\mathbb{P}(\phi(2 \min(X_1^{(0)}, X_2^{(0)})) \leq a) + \mathbb{P} \left(\phi \left(\frac{1}{2} \max(X_1^{(0)}, X_2^{(0)}) \right) \leq a \right) \right) \\
&= \frac{1}{2} (\mathbb{P}(\phi(\min(X_1^{(0)}, X_2^{(0)})) \leq a) + \mathbb{P}(\phi(\max(X_1^{(0)}, X_2^{(0)})) \leq a)) \\
&= \frac{1}{2} (\mathbb{P}(\phi(X_1^{(0)}) \leq a) + \mathbb{P}(\phi(X_2^{(0)}) \leq a)) \\
&= \mathbb{P}(\phi(X_1^{(0)}) \leq a)
\end{aligned}$$

□

Note that if X has distribution function F_0 , the fixed point already constructed, then the distribution of $\phi(X)$ is a point mass at $\frac{1}{2}$.

Corollary 5.6. *If we take a distribution function F satisfying $d(F, F_0) = C < \infty$ then we consider the space A_F of distribution functions G satisfying:*

1. $d(G, F_0) \leq C$.

2. If X_0 has distribution F and X_1 has distribution G , then $\phi(X_0)$ and $\phi(X_1)$ have the same distribution.
3. If X_0 has distribution F and X_1 has distribution G , then $\mathbb{E} \log(X_0) = \mathbb{E} \log(X_1)$.

then A_F contains a fixed point of γ .

Proof. The set A_F is compact with respect to d and is closed under γ , so A_F contains a fixed point of γ . \square

This gives the existence of a large family of fixed points.

5.5 Convergence

First consider the case of a log-symmetric (i.e. X has the same distribution as $1/X$) distribution concentrated on points of the form $2^n\sqrt{2}$ for $n \in \mathbb{Z}$. The constructed fixed point F_0 is of this form. If F is of this form, then the symmetry implies that

$$(F(2^n) - F_0(2^n))(F(2^{-n}) - F_0(2^{-n})) \leq 0 \text{ for all } n \in \mathbb{N}$$

and if $F \neq F_0$ then this quantity must be non-zero for some minimal $n(F) \geq 1$. Because of the effect of the map γ , if $n(F) \neq 1$ then $n(\gamma F) = n(F) - 1$, hence $n(\gamma^{n(F)-1} F) = 1$. But if $n(F) = 1$ then $d(\gamma F, F_0) < d(F, F_0)$ from (5.2). Hence $d(\gamma^n F, F_0)$ is a decreasing sequence, which is enough to prove convergence to F_0 .

However more generally convergence to a fixed point may not happen. If the initial distribution F is concentrated on points of the form $2^{2m}a$ then $\gamma^{2n}F$ will also be, while $\gamma^{2n+1}F$ will be concentrated on points of the form $2^{2m+1}a$ so no convergence can happen. Similar oscillating behaviour seems likely in other cases as well when $X_1^{(0)}$ is more likely to be near points of one of these forms than the other. So in general the limiting behaviour may

not be convergence to a fixed point. Behaviour such as a convergence to a periodic point or even some more complex behaviour may occur.

Chapter 6

Summary of results

The random hierarchical systems discussed show several different types of limiting behaviour. In two cases, the ‘length’ random variable of section 4 and the min-max system of section 5, we have proved the existence of non-degenerate fixed points of the map defined by the hierarchical system. In the latter case we can show that a certain class of initial distributions will converge to the fixed point but also we can see that some other initial distributions will display more complicated behaviour.

For the hierarchical systems describing the resistance of random fractal networks, we have three different types of behaviour. Convergence to a point mass at zero or infinity occurs for many parameter values. This appears to be usually rapid enough (in fact exponential) to imply almost sure convergence of individual random variables. For network 3.3.3 in section 3.3 we have convergence to a distribution with atoms at each of zero and infinity for a range of parameters. These results are obtained by a combination of bounding processes with branching processes, zero-one laws based on the network structure, and exponential decay of the mean resistance.

However for many parameter values ($p = \frac{1}{2}$ for the series-parallel network, and $p \in [\frac{1}{7}, \frac{1}{3}]$ for network 3.3.2 of section 3.3, for example) the questions of the existence of non-degenerate fixed points and convergence to them remain unsolved.

Chapter 7

Simulation

7.1 Introduction

The following work has been done using a series of Java applets to simulate the hierarchical systems. Due to the rapid growth of the number of random variables needed to calculate to stage n we can only do this for relatively small values of n .

7.2 Growth rates

We consider the series-parallel network of Chapter 3, and we take $p \geq \frac{1}{2}$. In this case the branching process analysis of Section 3.1 shows that $\liminf_{n \rightarrow \infty} (R^{(n)})^{\frac{1}{n}} \geq 2p$ almost surely, while the recurrence arguments on the mean (Section 3.2) show that $\limsup_{n \rightarrow \infty} (R^{(n)})^{\frac{1}{n}} \leq \frac{3p+1}{2}$ almost surely. We investigated the actual distribution of $(R^{(n)})^{\frac{1}{n}}$ for $p = 0.6, 0.7, 0.8$ and 0.9 , taking samples at $n = 10$ and $n = 20$.

p	Lower bound	Upper bound	Estimate $n = 10$	Estimate $n = 20$
0.6	1.2	1.4	1.323	1.316
0.7	1.4	1.55	1.515	1.513
0.8	1.6	1.7	1.680	1.685
0.9	1.8	1.85	1.848	1.846

7.3 Increase of variances

In section 3.2.4 we considered a variation on the series-parallel network with functions

$$f_1(x_1, x_2) = \mu(x_1 + x_2)$$

and

$$f_2(x_1, x_2) = \frac{x_1 x_2}{\mu(x_1 + x_2)}.$$

For $\frac{1}{2} < \mu < 1$ this is intermediate between the convergence to a point mass at $\mu = \frac{1}{2}$ (see Chapter 2) and the series-parallel network at $\mu = 1$ where $\text{Var} \log X_1^{(n)} \rightarrow \infty$ (Theorem 3.10) for $p = \frac{1}{2}$. In this case the variance increment can be negative.

Estimates of $\text{Var} \log X_1^{(n)}$ (for $p = \frac{1}{2}$) were obtained using samples of size 500 for four values of μ in this range, and also $\mu = 1$ for comparison. The estimates were:

μ	$n = 5$	$n = 10$	$n = 15$	$n = 20$
0.6	0.077	0.082	0.084	0.081
0.7	0.307	0.377	0.393	0.387
0.8	0.735	0.964	1.102	1.096
0.9	1.248	1.985	2.473	2.908
1	1.978	3.646	4.312	6.092

The difference in behaviour here is quite noticeable. It appears that the variance may be converging for the smaller values of μ whereas we know that it cannot for $\mu = 1$.

7.4 The three-resistor networks

For the resistance of the random fractal networks 3.3.2 and 3.3.4 from section 3.3 there are gaps between the range of p for which convergence to 0 has been proved and that for which convergence to ∞ has been proved. These were investigated using the simulation.

First, the behaviour of $\text{Var} \log X_1^{(n)}$ was investigated for network 3.3.4 and $p = \frac{1}{2}$. This is a log-symmetric system so we know that $\mathbb{E} \log X_1^{(n)} = 0$. The following are estimated variances for samples of size 500.

n	Estimated variance
2	0.253
4	0.366
6	0.353
8	0.404
10	0.427
12	0.380
14	0.419

This gives some evidence that the variance may not be converging to ∞ .

Next, for $p \in [\frac{2}{5}, \frac{1}{2})$, the behaviour of $\mathbb{E} \log X_1^{(n)}$ was investigated. The following are estimated means based on samples of size 500.

p	$n = 3$	$n = 6$	$n = 9$	$n = 12$
0.42	-0.190	-0.446	-0.609	-0.858
0.44	-0.162	-0.271	-0.440	-0.575
0.46	-0.101	-0.201	-0.334	-0.468
0.48	-0.026	-0.076	-0.091	-0.201

This suggests that this mean is converging to $-\infty$ at a fairly steady rate for each of these values of p , although larger values of n would give a clearer indication.

Similarly, for network 3.3.2, the behaviour of $\mathbb{E} \log X_1^{(n)}$ was investigated for $p \in [\frac{1}{7}, \frac{1}{3}]$. A further question here is, if there is not a range of p with a non-degenerate limit, then where the transition from convergence to 0 and ∞ occurs. The following are estimated means based on samples of size 500. The values $p = 0.14$ and $p = 0.34$, where the behaviour is known, are included for comparison.

p	$n = 3$	$n = 6$	$n = 9$	$n = 12$
0.14	-0.615	-1.292	-1.923	-2.614
0.18	-0.443	-0.820	-1.447	-1.847
0.22	-0.225	-0.561	-0.691	-0.978
0.26	-0.062	-0.024	-0.009	-0.077
0.3	0.193	0.369	0.655	0.944
0.34	0.370	0.929	1.366	2.127

The evidence here is that the transition occurs close to $p = 0.26$. There is no clear evidence of any of the other sequences of mean converging to a finite limit.

7.5 The length of the network

Finally the behaviour of the hierarchical system studied in section 4 and shown there to have a fixed point for $p < \frac{1}{2}$ was simulated.

The tables give a rough estimate of the distributions of $K_1^{(n)}$ found (with 500 samples) for two values of p and $n = 5, 10, 15, 20$.

$p = 0.1$	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\mathbb{P}(K_j^{(n)} = 1)$	0.850	0.888	0.882	0.906
$\mathbb{P}(K_j^{(n)} = 2)$	0.114	0.092	0.094	0.076
$\mathbb{P}(K_j^{(n)} = 3)$	0.032	0.012	0.010	0.012
$\mathbb{P}(K_j^{(n)} = 4)$	0.002	0.006	0.012	0.006
$\mathbb{P}(K_j^{(n)} \geq 5)$	0.002	0.002	0.002	0.000

$p = 0.4$	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\mathbb{P}(K_j^{(n)} = 1)$	0.414	0.380	0.360	0.334
$\mathbb{P}(K_j^{(n)} = 2)$	0.180	0.124	0.166	0.134
$\mathbb{P}(K_j^{(n)} = 3)$	0.096	0.106	0.100	0.100
$\mathbb{P}(K_j^{(n)} = 4)$	0.038	0.082	0.074	0.062
$\mathbb{P}(5 \leq K_j^{(n)} \leq 10)$	0.208	0.160	0.142	0.166
$\mathbb{P}(11 \leq K_j^{(n)} \leq 20)$	0.032	0.104	0.092	0.124
$\mathbb{P}(20 \leq K_j^{(n)} \leq 50)$	0.000	0.044	0.054	0.060
$\mathbb{P}(K_j^{(n)} \geq 51)$	0.000	0.000	0.012	0.020

The convergence seems much quicker for $p = 0.1$ than for $p = 0.4$. The estimates for $\mathbb{P}(K = 1)$ fit with the branching process theory which says that this is the survival probability of the branching process $\tilde{C}^{(n)}$ from Definition 3.1 which is $(1 - 2p)/(1 - p)$ for $p < \frac{1}{2}$. These values are $\frac{8}{9}$ and $\frac{1}{3}$ respectively.

Part III

Spectra of Laplacians of random fractal graphs

Chapter 8

Dirichlet-Neumann eigenvalues in the series-parallel graph

8.1 Introduction

As an example of a random fractal graph, we will give a detailed analysis of the spectral properties of the series-parallel graph. We will use both the replacement and duplication constructions from Definition 1.3 and we will use the Laplacian defined in Definition 1.6.

First we consider the two deterministic limits. At $p = 0$ $G^{(n)}$ is just 2^n parallel edges between two vertices, so the eigenvalues are just 0 and 2. At $p = 1$ it is a path with 2^n edges. The eigenvalues are $1 - \cos \frac{\pi k}{2^n}$, (Example 1.4 of [11]).

Definition 8.1. *We will call eigenfunctions x , such that $x(i) = 0$ for boundary vertices i , Dirichlet-Neumann eigenfunctions.*

As such eigenfunctions must satisfy $(\mathcal{L}x)(i) = 0$ for boundary vertices i this is similar to satisfying a Neumann as well as a Dirichlet boundary condition, hence the name. This definition applies to graphs in general, not just the series-parallel graphs.

The following lemma provides a means of constructing Dirichlet-Neumann eigenfunctions.

Lemma 8.2. *Let G_0 be a connected graph with m vertices, including distinguished endpoints 1 and 2, and let G be the graph formed by defining G_1 and G_2 to be two identical copies of G_0 and connecting them in parallel by identifying their endpoints. Then the Laplacian of G has $m-2$ linearly independent Dirichlet-Neumann eigenfunctions.*

The associated eigenvalues are the eigenvalues of the Laplacian \mathcal{L}_G restricted to the set $\{2j : 2 \leq j \leq m-1\}$ of vertices in G_2 .

Proof. We label the vertices of G_0 $1, 2, \dots, m$. Then we label the vertices in G so that, for $j \geq 3$, vertex j in G_0 corresponds to vertex $2j-3$ in G_1 and vertex $2j-2$ in G_2 .

Now consider the Laplacian \mathcal{L}_G . For $2 \leq j, k \leq m-1$ we have

$$\begin{aligned}\mathcal{L}_G(1, 2j-1) &= \mathcal{L}_G(1, 2j) \\ \mathcal{L}_G(2, 2j-1) &= \mathcal{L}_G(2, 2j) \\ \mathcal{L}_G(2j-1, 2k-1) &= \mathcal{L}_G(2j, 2k) \\ \mathcal{L}_G(2j-1, 2k) &= \mathcal{L}_G(2j, 2k-1) = 0\end{aligned}$$

and consider functions x satisfying

$$\begin{aligned}x(1) &= x(2) = 0 \\ x(2j-1) &= -x(2j) \text{ for } 2 \leq j \leq m-1.\end{aligned}$$

Now

$$(\mathcal{L}_G x)(1) = \sum_{j=2}^{m-1} (\mathcal{L}_G(1, 2j-1)x(2j-1) + \mathcal{L}_G(1, 2j)x(2j)) = 0$$

and similarly $(\mathcal{L}_G x)(2) = 0$, while

$$\begin{aligned}(\mathcal{L}_G x)(2j-1) &= \sum_{k=2}^{m-1} \mathcal{L}_G(2j-1, 2k-1)x(2k-1) \\ &= -\sum_{k=2}^{m-1} \mathcal{L}_G(2j, 2k)x(2k) = (\mathcal{L}_G x)(2j).\end{aligned}$$

So the Laplacian \mathcal{L}_G preserves vectors of this form, which form a vector space of dimension $m - 2$, and it acts on them like the Laplacian restricted to the interior vertices of G_2 . As the Laplacian is symmetric, there are $m - 2$ linearly independent Dirichlet-Neumann eigenfunctions of the Laplacian.

□

If $G^{(n)}$ contains a subgraph G of this form, where the vertices of G other than the endpoints have no edges linking them to $G^{(n)} \setminus G$, then we can take a Dirichlet-Neumann eigenfunction x on G and extend it to a Dirichlet-Neumann eigenfunction \tilde{x} on $G^{(n)}$ by setting

$$\begin{aligned}\tilde{x}(v) &= x(v) \text{ for } v \in V(G) \\ \tilde{x}(v) &= 0 \text{ otherwise}\end{aligned}$$

This holds for any Dirichlet-Neumann eigenfunction on a subgraph.

Corollary 8.3. *If there are r identical graphs connected at the endpoints in this way, we have $(r - 1)(m - 2)$ linearly independent Dirichlet-Neumann eigenfunctions.*

Proof. Lemma 8.2 gives $m - 2$ eigenfunctions localised on each pair, and there are $r - 1$ linearly independent pairs. □

The subgraph in Figure 8.1 provides a Dirichlet-Neumann eigenfunction with eigenvalue 1, while that in Figure 8.2 provides two Dirichlet-Neumann eigenfunctions with eigenvalues $1 \pm \sqrt{1/6}$.

The first structure occurs as $G^{(2)}$ with probability $p^2(1 - p)$, while the second (or its reflection) occurs as $G^{(3)}$ with probability $2p^2(1 - p)^3$.

We consider the duplication sequence of graphs $G_j^{(n)}$, where $G_1^{(0)} = G^{(0)}$ and $G_1^{(n)}$ is obtained by connecting $G_1^{(n-1)}$ to an independent copy $G_2^{(n-1)}$, in series with probability p and in parallel with probability $1 - p$.

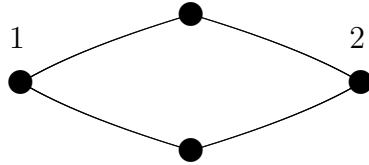


Figure 8.1: A subgraph responsible for Dirichlet-Neumann eigenfunctions with eigenvalue 1

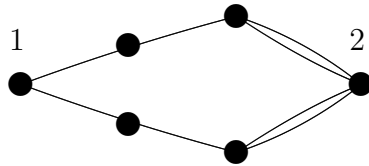


Figure 8.2: A subgraph responsible for Dirichlet-Neumann eigenfunctions with eigenvalues $1 \pm \frac{1}{\sqrt{6}}$

The advantage of working with the duplication sequence in this context is that the Dirichlet-Neumann eigenfunctions of $G_1^{(n-1)}$ and $G_2^{(n-1)}$ are preserved in $G_1^{(n)}$.

8.2 A lower bound for the multiplicity of one particular eigenvalue

We now consider, as an example, the structure in Figure 8.1. We set $A_1^{(n)}$ to be the multiplicity of the eigenvalue 1 arising from subgraphs of this form in $G_1^{(n)}$, and $B_1^{(n)}$ to be the multiplicity of the eigenvalue 1 arising from those subgraphs whose endpoints are the endpoints of the graph.

We also set $D_1^{(n)}$ to be the number of paths of two edges linking the two boundary points. So $B_1^{(n)} = D_1^{(n)} - I_{\{D_1^{(n)} \neq 0\}}$.

Now, conditional on $G_1^{(n)}$ having its input graphs connected in series

$$\begin{aligned} A_1^{(n)} &= A_1^{(n-1)} + A_2^{(n-1)} \\ B_1^{(n)} &= 0 \end{aligned}$$

while conditional on $G_1^{(n)}$ having its input graphs connected in parallel

$$A_1^{(n)} = (A_1^{(n-1)} - B_1^{(n-1)}) + (A_2^{(n-1)} - B_2^{(n-1)}) + B_1^{(n)}.$$

Now considering the branching process $\tilde{C}^{(n)}$ from Definition 3.1 we see that

$$\mathbb{E}D_1^{(n)} = p\mathbb{E}\tilde{C}^{(n-1)} = p(2 - 2p)^{n-1}$$

and so

$$\mathbb{E}B_1^{(n)} = p(2 - 2p)^{n-1} - q_n$$

where $q_n = \mathbb{P}(D_1^{(n)} \neq 0)$.

But $\mathbb{E}(B_1^{(n)} | G_1^{(n)} \text{ series}) = 0$, so

$$\mathbb{E}(B_1^{(n)} | G_1^{(n)} \text{ parallel}) = 2p(2 - 2p)^{n-2} - \frac{q_n}{1 - p}$$

and so

$$\begin{aligned}
\mathbb{E}(B_1^{(n)} | G_1^{(n)} \text{ parallel}) &= 2\mathbb{E}A_1^{(n-1)} - 2\mathbb{E}B_1^{(n-1)} + 2p(2-2p)^{n-2} - \frac{q_n}{1-p} \\
&= 2\mathbb{E}A_1^{(n-1)} + 2q_{n-1} - \frac{q_n}{1-p} \\
&= 2\mathbb{E}A_1^{(n-1)} + q_{n-1}^2,
\end{aligned}$$

because $q_n = (1-p)(2q_{n-1} - q_{n-1}^2)$ for $n \geq 2$ by the definition of q_n . Removing the conditioning gives $\mathbb{E}A_1^{(n)} = 2\mathbb{E}A_1^{(n-1)} + q_{n-1}^2(1-p)$ and hence

$$\mathbb{E} \frac{A_1^{(n)}}{2^n} = \sum_{i=2}^n \frac{1}{2^i} q_{i-1}^2 (1-p)$$

which gives a lower bound for the limit $\lim_{n \rightarrow \infty} \mathbb{E} \frac{l_{1,1}^{(n)}}{2^n}$ where $l_{1,1}^{(n)}$ is the multiplicity of Dirichlet-Neumann eigenfunctions with eigenvalue 1.

8.3 Near-localised eigenvalues

The replacement construction of the series-parallel graph has the property that the degree of vertices grows exponentially in n , because it follows a branching process with offspring distribution concentrated on $\{1, 2\}$.

This makes the following lemma relevant:

Lemma 8.4. *Consider a fixed graph G_0 , with defined boundary vertices 1 and 2, and a graph G whose vertex set can be partitioned as $\{1\} \cup \{2\} \cup S \cup R$, where there are no edges between S and R , and the subgraph defined on $\{1\} \cup \{2\} \cup S$ is isomorphic to G_0 with the boundary preserved.*

We let $\{\lambda_i : 1 \leq i \leq |S|\}$ be the Dirichlet-Neumann eigenvalues obtained in Lemma 8.2 for two copies of G_0 with boundary vertices identified, and we let δ_1 and δ_2 be the degrees of the boundary vertices.

Then, for any ϵ , there will be an eigenvalue λ_i^ of G such that $\lambda_i^* \in (\lambda_i - \epsilon, \lambda_i + \epsilon)$, assuming both δ_1 and δ_2 are sufficiently large.*

Proof. Without loss of generality we label the vertices such that $S = [3, |S + 2|]$ and that the vertices have the same labels as in G_0 .

We consider the Laplacian \mathcal{L}_G of G . For $s \in S, r \in R$,

$$\mathcal{L}_G(r, s) = \mathcal{L}_G(s, r) = 0,$$

while

$$\mathcal{L}_G(1, s) = \mathcal{L}_G(s, 1) = \frac{-w(1, r)}{\sqrt{\delta_1 \delta_r}}$$

and similarly

$$\mathcal{L}_G(2, s) = \mathcal{L}_G(s, 2) = \frac{-w(2, r)}{\sqrt{\delta_2 \delta_r}}.$$

But note that $m(1, s) = \frac{-w(1, s)}{\sqrt{\delta_s}}$ and $m(2, s) = \frac{-w(2, s)}{\sqrt{\delta_s}}$ depend only on the structure of G_0 and hence are constants.

Hence

$$\mathcal{L}_G(1, s) = \mathcal{L}_G(s, 1) = \frac{m(1, s)}{\sqrt{\delta_1}}$$

and similarly

$$\mathcal{L}_G(2, s) = \mathcal{L}_G(s, 2) = \frac{m(2, s)}{\sqrt{\delta_2}}. \quad (8.1)$$

So we now consider the eigenvalues of the matrix \mathcal{L}' with

$$\mathcal{L}'(1, s) = \mathcal{L}'(2, s) = \mathcal{L}'(s, 1) = \mathcal{L}'(s, 2) = 0$$

for each $s \in S$, and $\mathcal{L}'(i, j) = \mathcal{L}_G(i, j)$ otherwise.

Both \mathcal{L}_G and \mathcal{L}' are symmetric matrices, so we can use the perturbation theorem (2.44.9) of [53], which tells us that for each eigenvalue α_i of \mathcal{L}' there will be an eigenvalue γ_i of \mathcal{L}_G so that $\gamma_i \in [\alpha_i + \beta_1, \alpha_i + \beta_n]$, where β_1 and β_n are respectively the smallest and largest eigenvalues of $\mathcal{L}_G - \mathcal{L}'$.

But

$$|\beta_1| \leq \frac{B}{\sqrt{\min\{\delta_1, \delta_2\}}}$$

and similarly for $|\beta_n|$, where B is a constant depending only on G_0 , deriving from (8.1).

Finally, the eigenvalues of \mathcal{L}' contain each λ_i , because the block $\{\mathcal{L}_G(s_1, s_2) : s_1, s_2 \in S\}$ is the same matrix as that constructed in Lemma 8.2, and the other entries in these rows and columns are zero.

□

In the context of the series-parallel graph $G^{(n)}$, the subgraph S isomorphic to G_0 is an m -cell whose boundary vertices have high multiplicity.

8.4 Simulation of eigenvalues for the series-parallel network

Eigenvalues of samples of the non-homogeneous series parallel network were taken using a Java program to construct the graphs and Maple VI to find the eigenvalues. The plots are histograms showing the different eigenvalues found for a sample of graphs.

The concentrations that appear on particular values (especially 1) appear to be due to Dirichlet-Neumann and near-localised eigenfunctions. It is noticeable that these concentrations seem to form a greater part of the spectrum for smaller p . The distribution of the non-localised part of the spectrum is also interesting - for $p = 0.5$ it seems to be close to uniform, and for $p = 0.9$ it is concentrated towards 0 and 2 (compare the eigenvalues $1 - \cos \frac{\pi k}{n-1}$ for the path on n vertices [11], which is the special case $p = 1$.)

The eigenvalues of these random graphs show quite different distributions to the semi-circle law found for Erdős-Rényi random graphs, and also those found for small world and scale-free graphs discussed in [18].

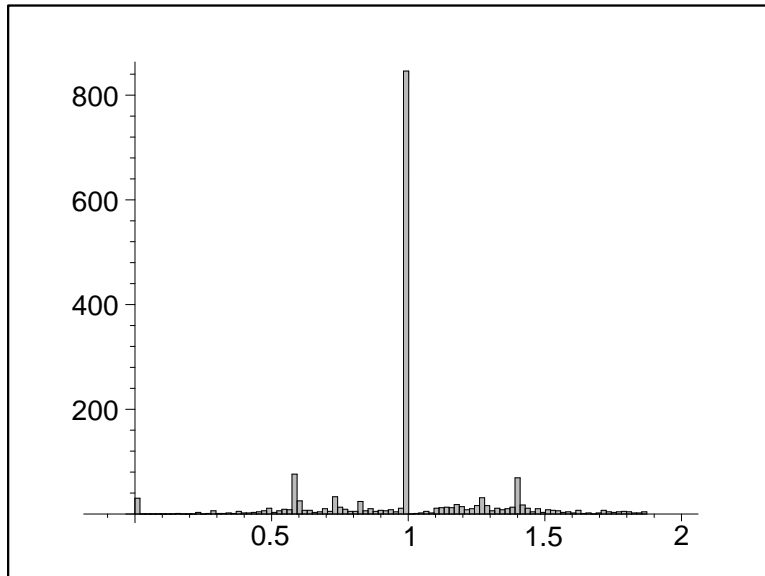


Figure 8.3: Histogram of eigenvalues of $G^{(9)}$ with $p = 0.1$, 30 graphs sampled

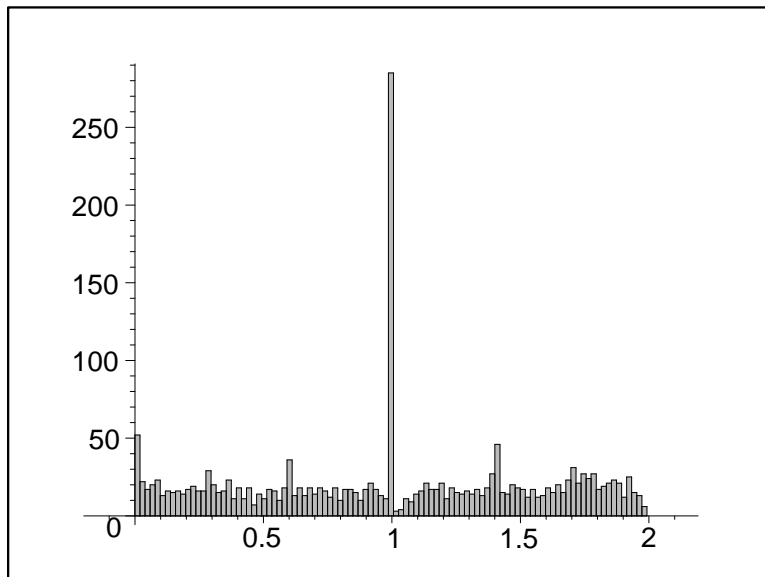


Figure 8.4: Histogram of eigenvalues of $G^{(7)}$ with $p = 0.5$, 30 graphs sampled

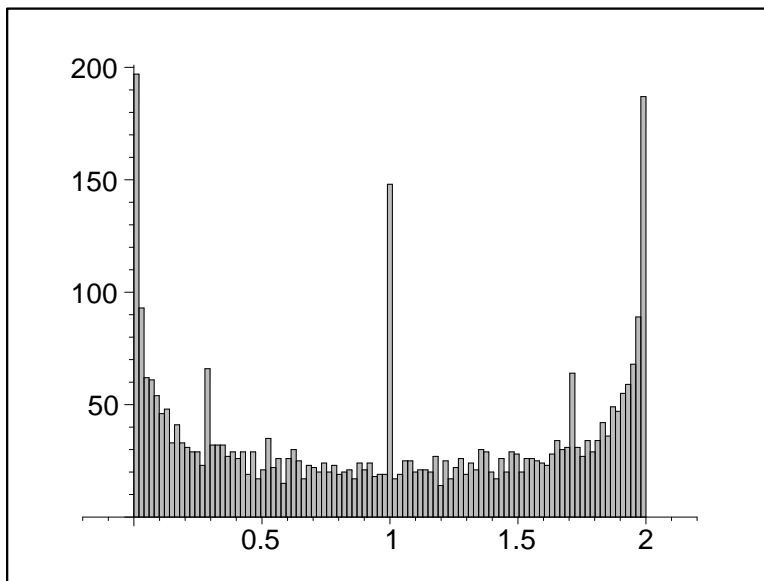


Figure 8.5: Histogram of eigenvalues of $G^{(7)}$ with $p = 0.9$, 30 graphs sampled

8.5 Cheeger constants

For a subset $S \subseteq G^{(n)}$, following [11], we define $\text{vol } S$ to be the sum of the degrees of vertices in S , and define \bar{S} to be the complement of S in $V(G^{(n)})$. Further we define $E(S_1, S_2)$ to be the number of edges between vertices in S_1 and vertices in S_2 .

We now define

$$h_{G^{(n)}}(S) = \frac{|E(S, \bar{S})|}{\min\{\text{vol } S, \text{vol } \bar{S}\}}$$

and the *Cheeger constant*

$$h_{G^{(n)}} = \min_S h_{G^{(n)}}(S).$$

This is a measure of the connectivity of the graph, see [11] for further details. Where $\lambda_1(G^{(n)})$ is the smallest positive eigenvalue of $G^{(n)}$, we have the

Cheeger inequality (proved in [11]):

$$2h_{G^{(n)}} \geq \lambda_1(G^{(n)}) > \frac{h_{G^{(n)}}^2}{2}.$$

So bounds on the Cheeger constant can give bounds on the lowest positive eigenvalue. We consider the asymptotics of the Cheeger constants of the series-parallel graphs $G^{(n)}$.

Lemma 8.5. *The Cheeger constant satisfies*

$$\frac{1}{n} \log h_{G^{(n)}} \geq \log \frac{1}{2}.$$

Furthermore, if $p > \frac{1}{2}$, this bound is asymptotically tight, i.e.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log h_{G^{(n)}} = \log \frac{1}{2}.$$

Proof. As $G^{(n)}$ is connected, $|E(S, \bar{S})| \geq 1$ for any subset $S \subseteq V(G^{(n)})$. Furthermore, $\text{vol } G^{(n)} = 2^{n+1}$ as there are 2^n edges in $G^{(n)}$. So $\min\{\text{vol } S, \text{vol } \bar{S}\} \leq 2^n$, giving the lower bound.

To prove that it is tight, we consider the branching process $C^{(n)}$ from Definition 3.1, which is the number of 1-cuts (i.e. edges whose removal would disconnect the graph). This is supercritical when $p > \frac{1}{2}$ so with positive probability $C^{(n)} \rightarrow \infty$. So if we have a vertex set $S^{(n)} \subseteq V(G^{(n)})$, there is positive probability of the existence of descendant sets $S^{(m)} \subseteq V(G^{(m)})$, $m > n$, such that $S^{(n)} \subseteq S^{(m)}$, $\bar{S}^{(n)} \subseteq \bar{S}^{(m)}$ and $|\partial S^{(m)}| = |\partial S^{(n)}|$ for all m . If $S^{(n)}$ and $\bar{S}^{(n)}$ both contain an interior edge then $\text{vol } S^{(m)}$ and $\text{vol } \bar{S}^{(m)}$ will both be at least $\epsilon 2^m$ for some $\epsilon > 0$ so this is enough to show that there is positive probability that $\lim_{n \rightarrow \infty} \frac{1}{n} \log h_{G^{(n)}} = \log \frac{1}{2}$. Now note that the existence of a set $S^{(n)}$ (of any one of the sets $G^{(n)}$) with a sequence of $S^{(m)}$, $m > n$ satisfying the above property is an event of the same type as convergence of $R^{(n)}$ to ∞ ; so by the proof of Lemma 3.3 its probability must be 0 or 1. This gives the result. \square

The intuition here is that when $p > \frac{1}{2}$ the structure looks like a relatively small number of long strands, so it is easy to make small cuts. We do not expect this to be true for small p where the graph should be more connected.

The following gives an upper bound on the Cheeger constant when $p \leq \frac{1}{2}$:

Lemma 8.6. *The sequence of Cheeger constants satisfies*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log h_{G^{(n)}} \leq \log \frac{2-p}{2}$$

almost surely.

Proof. Construct a sequence of sets of vertices $S^{(n)} \subseteq V(G^{(n)}); n \in \mathbb{N}$ as follows: At stage 0 put vertex $1 \in S^{(0)}$, vertex $2 \in \bar{S}^{(0)}$. Then at stage $n+1$ consider each edge of $G^{(n)}$ in $E(S^{(n)}, \bar{S}^{(n)})$. The edge will be $a \leftrightarrow b$ with $a \in S^{(n)}$ and $b \in \bar{S}^{(n)}$. If the edge is replaced by a pair of edges in parallel $a \leftrightarrow b$ then keep $a \in S^{(n+1)}$ and $b \in \bar{S}^{(n+1)}$. If the edge is replaced with a pair of edges in series $a \leftrightarrow c$ and $c \leftrightarrow b$ then keep $a \in S^{(n)}$ and $b \in \bar{S}^{(n+1)}$, and place the new vertex $c \in S^{(n)}$ with probability $\frac{1}{2}$.

This ensures that $\frac{1}{n} \log(\text{vol } S^{(n)}) \rightarrow \log 2$ and $\frac{1}{n} \log(\text{vol } \bar{S}^{(n)}) \rightarrow \log 2$, almost surely, and further that $|E(S^{(n)}, \bar{S}^{(n)})|$ behaves as a branching process with offspring distribution 1 with probability p , 2 with probability $1-p$, and hence $\frac{1}{n} \log |E(S^{(n)}, \bar{S}^{(n)})| \rightarrow \log(2-p)$ almost surely. This gives the result. \square

Lemma 8.7. *The sequence of Cheeger constants satisfies*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log h_{G^{(n)}} \geq -2p \log 2.$$

Proof. We let $S^{(n)}$ be a subset of the vertex set such that

$$h_{G^{(n)}} = \frac{|E(S^{(n)}, \bar{S}^{(n)})|}{\text{vol } \bar{S}^{(n)}}.$$

Note that we may assume that $\bar{S}^{(n)}$ forms a connected subgraph - this is because $\frac{a+b}{c+d} \geq \min(\frac{a}{c}, \frac{b}{d})$ for positive a, b, c, d so a disconnected $\bar{S}^{(n)}$ will always contain a connected component giving at least as good a Cheeger constant.

We define $k^{(n)} = \min\{a \in \bar{S}^{(n)}\}$ and $\hat{k}^{(n)}$ to be its neighbour in $G^{(j(k^{(n)}))}$, i.e. $n(k^{(n)})$. We consider the evolution of the edges replacing the edge $\hat{k}^{(n)} \leftrightarrow k^{(n)}$ in $G^{(j(k^{(n)}))}$. (Note that no vertex c with $j(c) < j(k^{(n)})$ can be in $\bar{S}^{(n)}$.)

The resistance $R^{(n)}(\hat{k}^{(n)}, k^{(n)})$ between $\hat{k}^{(n)}$ and $k^{(n)}$ in $G^{(n)}$ can be bounded below by the reciprocal of the minimum cut between $\hat{k}^{(n)}$ and $k^{(n)}$, which is at least $\frac{1}{|E(S^{(n)}, \bar{S}^{(n)})|}$. So

$$|E(S^{(n)}, \bar{S}^{(n)})| \geq \frac{1}{R^{(n)}(\hat{k}^{(n)}, k^{(n)})}.$$

Because of the connectivity of $\bar{S}^{(n)}$ and the graph structure, the volume satisfies

$$\text{vol } \bar{S}^{(n)} \leq 2^{n-j(k^{(n)})+1}$$

(any volume larger than this containing $k^{(n)}$ will contain one of the two original (i.e. in $G^{(j(k^{(n)}))}$) neighbours of $k^{(n)}$, which are not in $\bar{S}^{(n)}$ by hypothesis) so

$$h_{G^{(n)}} \geq \frac{1}{2^{n-j(k^{(n)})+1} R^{(n)}(\hat{k}^{(n)}, k^{(n)})}.$$

Now

$$R^{(n)}(\hat{k}^{(n)}, k^{(n)}) =_d R_n^{(n-j(k^{(n)}))} =_d X_1^{(n-j(k^{(n)}))}$$

in the hierarchical model. So we use the complete convergence results of Lemma 3.18 and Corollary 3.21 as follows:

$$\begin{aligned} h_{G^{(n)}} &\geq_d \frac{1}{2^{n-j(k^{(n)})+1} X_1^{(n-j(k^{(n)}))}} \\ 2(2\mu)^{n-j(k^{(n)})} h_{G^{(n)}} &\geq_d \frac{1}{\mu^{-(n-j(k^{(n)})+1)} X_1^{(n-j(k^{(n)}))}}. \end{aligned}$$

Using Corollary 3.21 we choose $\mu > 2^{2p-1}$ so that

$$\frac{1}{\mu^{-(n-j(k^{(n)})+1)} X_1^{(n-j(k^{(n)}))}} \rightarrow \infty \text{ a.s.}$$

as long as $n - j(k^{(n)}) \rightarrow \infty$ a.s. This condition is satisfied because of the bound above on the volume and the lim sup result for the Cheeger constant in Lemma 8.6.

So, for $\mu > 2^{2p-1}$,

$$(2\mu)^n h_{G^{(n)}} \geq (2\mu)^{n-j(k^{(n)})} h_{G^{(n)}} \rightarrow \infty$$

almost surely, because of the complete convergence in Lemma 3.18.

So $\log 2\mu + \liminf_{n \rightarrow \infty} \frac{1}{n} \log h_{G^{(n)}} \geq 0$, which gives the result. \square

We now put Lemmas 8.5, 8.6 and 8.7 together to obtain the following:

Theorem 8.8. *When $p > \frac{1}{2}$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log h_{G^{(n)}} = -\log 2$$

almost surely.

When $p \leq \frac{1}{2}$,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log h_{G^{(n)}} \geq -2p \log 2$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log h_{G^{(n)}} \leq \log \frac{2-p}{2},$$

also almost surely.

This gives a clear transition in behaviour as p passes through $\frac{1}{2}$.

Corollary 8.9. *When $p > \frac{1}{2}$, the lowest positive eigenvalue $\lambda_1(G^{(n)})$ satisfies*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda_1(G^{(n)}) \geq -\frac{1}{2} \log 2$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda_1(G^{(n)}) \leq -\log 2.$$

When $p < \frac{1}{2}$, it satisfies

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda_1(G^{(n)}) \geq -p \log 2$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda_1(G^{(n)}) \leq \log \frac{2-p}{2}.$$

Proof. This is immediate from Theorem 8.8 and the Cheeger inequality. \square

Note that the large spectral gap predicted by this result for small p can be seen in the simulation results for $p = 0.1$, seen in Figure 8.3.

Chapter 9

Eigenvalues of homogeneous random fractal graphs

9.1 Eigenvalues of the homogeneous series-parallel network

We first consider some properties of the homogeneous version of the series-parallel network. As in section 1.2 F_n will be the stage n replacement graph and F'_n will be the duplication graph.

It is easy to show that the resistance R_n of F_n is 2^{X_n} where $(X_n)_{n \in \mathbb{N}}$ is a simple random walk on \mathbb{R} with birth probability p and death probability $1 - p$. Hence we see that for $p < \frac{1}{2}$, $R_n \rightarrow 0$ almost surely and for $p > \frac{1}{2}$, $R_n \rightarrow \infty$ almost surely, while when $p = \frac{1}{2}$ we will have $\liminf_{n \rightarrow \infty} R_n = 0$ and $\limsup_{n \rightarrow \infty} R_n = \infty$.

However the branching process methods applied to the non-homogeneous case do not work here at all - with probability 1 there will be no edges connecting the endpoints or whose removal disconnects F_n for n large enough.

We define \mathcal{L}_n to be the Laplacian of F'_n , and v_n to be the total number

of eigenvalues of F'_n (including multiplicity) - this is obviously equal to the number of vertices of F'_n .

We now let l_n be the number of linearly independent Dirichlet-Neumann eigenfunctions of F'_n , i.e. those which are zero on the boundary vertices.

The following lemmas will be useful:

Lemma 9.1. *If we take two identical copies G_1 and G_2 of a graph G and obtain a new graph H by identifying some pairs of corresponding vertices, then the eigenvalues of G are included in the eigenvalues of H .*

Proof. We label the vertices so that vertices corresponding to vertex j in G are vertices $2j - 1$ in G_1 and $2j$ in G_2 , for $1 \leq j \leq k_1$, the number of vertices of G . We then identify vertices $2j$ and $2j - 1$ for $j < k_0$, so H has no vertex $2j$, $j < k_0$.

We will consider the matrix M defined by $M(j, j) = 1$ and

$$M(j, k) = -\frac{e(j, k)}{d(j)}$$

where $e(j, k)$ is the number of edges in G linking vertices j and k and $d(j)$ is the degree of vertex j in G . Then $M = T^{-\frac{1}{2}}\mathcal{L}T^{\frac{1}{2}}$ (where T is the diagonal matrix whose entries are the degrees of the vertices) and so M and \mathcal{L} have the same eigenvalues. Note that $-M$ is the generator of a continuous time random walk on the graph and both M and $-M$ may also be referred to as Laplacians.

Let M' be the corresponding matrix for H . Then for $k, j \leq k_0$,

$$M'(2k - 1, 2j - 1) = M(k, j) \text{ for } k, j \leq k_0$$

and for $k \leq k_0$ but $j > k_0$,

$$M'(2k - 1, 2j - 1) = M'(2k - 1, 2j) = \frac{1}{2}M(k, j)$$

$$M'(2j - 1, 2k - 1) = M'(2j, 2k - 1) = M(j, k).$$

For $j, k > k_0$,

$$\begin{aligned} M'(2j-1, 2k) &= M'(2k, 2j-1) = 0 \\ M'(2j-1, 2k-1) &= M'(2j, 2k) = M(j, k). \end{aligned}$$

We now take an eigenfunction x (with eigenvalue λ) of M and define a new function x' by

$$\begin{aligned} x'(2j-1) &= x(j) \text{ for all } j \leq k_1 \\ x'(2j) &= x(j) \text{ for all } j \text{ with } k_0 < j \leq k_1. \end{aligned}$$

For $k \leq k_0$,

$$\begin{aligned} (M'x')(2k-1) &= \sum_{j=1}^{k_0} M(k, j)x(j) + \sum_{j=k_0+1}^{k_1} 2\frac{1}{2}M(k, j)x(j) \\ &= \sum_{j=1}^{k_1} M(k, j)x(j) = \lambda x(k) \end{aligned}$$

and for $k > k_0$,

$$\begin{aligned} (M'x')(2k) &= (M'x')(2k-1) \\ &= \sum_{j=1}^{k_0} M(k, j)x(j) + \sum_{j=k_0+1}^{k_1} M(k, j)x(j) \\ &= \sum_{j=1}^{k_1} M(k, j)x(j) = \lambda x(k). \end{aligned}$$

So x' is an eigenfunction of M' with eigenvalue λ and hence λ is an eigenvalue of H . So the spectrum of G is included in that of H . \square

This enables us to obtain the following results for the homogeneous series-parallel network (using the duplication construction):

Lemma 9.2. *Conditional on F'_n being formed by a parallel connection, then*

$$\begin{aligned} v_n &= 2v_{n-1} - 2 \\ l_n &\geq l_{n-1} + v_{n-1} - 2 \end{aligned}$$

Proof. The vertex set of F'_n is the union of two sets $V_{n-1}^{(1)}$ and $V_{n-1}^{(2)}$, which represent the vertex sets of the copies of F'_{n-1} . The intersection of these sets is the boundary $\{1, 2\}$. We label the remaining vertices in such a way that the two vertices corresponding to vertex j of F'_{n-1} ($j \geq 3$) are vertices $2(j-1) - 1 \in V_{n-1}^{(1)}$ and $2(j-1) \in V_{n-1}^{(2)}$.

This implies that \mathcal{L}_n is obtained from \mathcal{L}_{n-1} in such a way that

$$\begin{aligned}
\mathcal{L}(2(j-1) - 1, 2(k-1)) &= 0 \text{ for all } 2 \leq j, k \leq v_{n-1} - 1 \\
\mathcal{L}_n(2j-1, 2k-1) &= \mathcal{L}_n(2j, 2k) \\
&= \mathcal{L}_{n-1}(j+1, k+1) \text{ for all } 2 \leq j, k \leq v_{n-1} - 1 \\
\mathcal{L}_n(1, 2j-1) &= \mathcal{L}_n(1, 2j) \\
&= \sqrt{\frac{1}{2}} \mathcal{L}_{n-1}(1, j+1) \text{ for all } 2 \leq j \leq v_{n-1} - 1 \\
\mathcal{L}_n(2, 2j-1) &= \mathcal{L}_n(2, 2j) \\
&= \sqrt{\frac{1}{2}} \mathcal{L}_{n-1}(2, j+1) \text{ for all } 2 \leq j \leq v_{n-1} - 1.
\end{aligned}$$

The Laplacians used are symmetric and have ones on the diagonal, so this defines \mathcal{L}_n .

We now consider an Dirichlet-Neumann eigenfunction x of F'_{n-1} . We extend this to an Dirichlet-Neumann eigenfunction \hat{x} for F'_n by setting $\hat{x}(2(j-1) - 1) = x(j)$ and $\hat{x}(2(j-1)) = 0$ for $3 \leq j \leq v_{n-1}$, and $\hat{x}(1) = \hat{x}(2) = 0$. This provides l_{n-1} Dirichlet-Neumann eigenfunctions for F'_n .

Now we use Lemma 8.2 to obtain $v_{n-1} - 2$ linearly independent Dirichlet-Neumann eigenfunctions satisfying $x(1) = x(2) = 0$, and $x(2j-1) = x(2j)$ for all j , which furthermore are linearly independent of those already found. So at least $l_{n-1} + v_{n-1} - 2$ can be found. \square

Lemma 9.3. *Conditional on F'_n being formed by a series connection, then*

$$\begin{aligned}
v_n &= 2v_{n-1} - 1 \\
l_n &\geq 2l_{n-1}
\end{aligned}$$

Proof. The proof is similar to above, but the two sets of linearly independent Dirichlet-Neumann eigenfunctions are derived directly from the Dirichlet-

Neumann eigenfunctions of F'_{n-1} , each of which gives one linearly independent eigenfunction Dirichlet-Neumann on each of the vertex subsets corresponding to the input graphs. \square

Theorem 9.4. *The ratio l_n/v_n converges to 1 as $n \rightarrow \infty$, almost surely.*

Proof. We will define \hat{l}_n to be the number of Dirichlet-Neumann eigenvalues of the graph F'_n which are constructed in the proofs of Lemmas 9.2 and 9.3.

We consider the difference

$$\frac{\hat{l}_n}{v_n} - \frac{\hat{l}_{n-1}}{v_{n-1}}$$

When the connection is series,

$$\begin{aligned} \frac{\hat{l}_n}{v_n} - \frac{\hat{l}_{n-1}}{v_{n-1}} &= \frac{2v_{n-1}\hat{l}_{n-1} - (2v_{n-1} - 1)\hat{l}_{n-1}}{v_{n-1}(2v_{n-1} - 1)} \\ &= \frac{\hat{l}_{n-1}}{v_{n-1}(2v_{n-1} - 1)} \end{aligned}$$

and when it is in parallel

$$\begin{aligned} \frac{\hat{l}_n}{v_n} - \frac{\hat{l}_{n-1}}{v_{n-1}} &= \frac{(\hat{l}_{n-1} + v_{n-1} - 2)v_{n-1} - (2v_{n-1} - 2)\hat{l}_{n-1}}{v_{n-1}(2v_{n-1} - 2)} \\ &= \frac{v_{n-1}^2 - v_{n-1} - \hat{l}_{n-1}v_{n-1} + \hat{l}_{n-1} + (\hat{l}_{n-1} - v_{n-1})}{2v_{n-1}(v_{n-1} - 1)} \\ &= \frac{1}{2} \left(1 - \frac{\hat{l}_{n-1}}{v_{n-1}} \right) - \frac{v_{n-1} - \hat{l}_{n-1}}{2v_{n-1}(v_{n-1} - 1)} \\ &\geq \frac{1}{2} \left(1 - \frac{\hat{l}_{n-1}}{v_{n-1}} \right) - \frac{1}{2v_{n-1}} \end{aligned}$$

So the ratio is strictly increasing in n . If we assume that the limit is $q < 1$, then for n sufficiently large (because $v_n \rightarrow \infty$), the increments for the parallel case will be at least $\frac{1}{2}(1 - q) - \epsilon$ for any $\epsilon > 0$. As there will almost surely be infinitely many of these increments (for $p < 1$) the limit must be 1.

As (by definition) $\hat{l}_n \leq l_n$, this completes the proof.

□

We define $l_{n,\lambda}$ to be the Dirichlet-Neumann multiplicity of λ in F'_n , i.e. the number of linearly independent Dirichlet-Neumann eigenfunctions with eigenvalue λ .

Corollary 9.5. *The ratio*

$$\frac{l_{n,\lambda}}{v_n} \uparrow l_{\infty,\lambda} \text{ as } n \rightarrow \infty,$$

for some random variable $l_{\infty,\lambda}$. This works for any homogeneous duplication sequence as defined in Definition 1.3 under Assumption 1.1.

Proof. As in Lemma 9.3, given a Dirichlet-Neumann eigenfunction of F'_{n-1} , we can obtain one Dirichlet-Neumann eigenfunction localised on each of the $(n-1)$ -cells in F'_n , with the same eigenvalue λ . So $l_{n,\lambda} \geq l_{n-1,\lambda}c(Z_n)$ (where $c(Z_n)$ is the number of $(n-1)$ -cells in F'_n). But $v_n \leq v_{n-1}c(Z_n)$, so $(\frac{l_{n,\lambda}}{v_n})_{n \in \mathbb{N}}$ is an increasing sequence in $[0, 1]$, giving the result. □

This shows that, for a homogeneous duplication sequence where $\lim_{n \rightarrow \infty} \frac{l_n}{v_n} = 1$, the sequence of spectral measures of F'_n converges to a pure point limit.

9.2 Spectral similarity

We will discuss the use of the technique of *spectral similarity* (or *spectral decimation*), which provides information about the eigenvalues of deterministic fractals, as described in [35, 36, 46, 50], to obtain similar results for some homogeneous random fractal graphs.

We follow [36]. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be Hilbert spaces, and let H be an operator on \mathcal{H} and H_0 be an operator on \mathcal{H}_0 . We let P_0 be the orthogonal projector from \mathcal{H} onto \mathcal{H}_0 .

If ϕ_0 and ϕ_1 are complex valued functions defined on a set $\Lambda \in \mathbb{C}$, and

$$P_0(H - z)^{-1} = (\phi_0(z)H_0 - \phi_1(z))^{-1}$$

for all $z \in \Lambda$ where the expression is well-defined, then we say that H is spectrally similar to H_0 with functions ϕ_0 and ϕ_1 .

We set $R(z) = \frac{\phi_1(z)}{\phi_0(z)}$.

We now let \mathcal{H}_1 be the orthogonal complement to \mathcal{H}_0 and consider H in a block structure with respect to the representation $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$:

$$H = \begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix}.$$

We define (Definition 3.5 of [36]) the *exceptional set* $\mathcal{E}(H, H_0)$ of H and H_0 to be $\{z : z \text{ an eigenvalue of } Q \text{ or } R(z) \text{ undefined}\}$.

Then Theorem 3.6 of [36] says that, if H is spectrally similar to H_0 and $z \notin \mathcal{E}(H, H_0)$, then z is an eigenvalue of H if and only if $R(z)$ is an eigenvalue of H_0 , and further that the respective multiplicities of z and $R(z)$ are the same.

Spectral similarity applies only to the replacement constructions of some homogeneous random fractals. It does not apply for deterministic fractal graphs without enough symmetry, such as the pentagasket (see [1]) and also will not apply to related homogeneous random fractal graphs. However, if the model graphs are symmetric enough, then the method can be used to describe the limiting spectrum. This is done for the Sierpiński gasket in [46, 50].

Assuming that the initial cell F_0 is a complete graph, and that the model graph Γ_n is symmetric with respect to F_0 as defined in Definition 4.1 of [36] (i.e. any bijection $F_0 \rightarrow F_0$ can be extended to a graph automorphism $\Gamma_n \rightarrow \Gamma_n$), then we can apply Lemma 4.7 of [36] to obtain the spectral similarity of the Laplacian of F_n to that of F_{n-1} .

As an example of how this applies to homogeneous random fractal graphs we now return to the homogeneous series-parallel network. The spectral

similarity results do not apply directly but there is a closely related sequence of homogeneous graphs to which they do apply. We consider the subsequence $(F_{a(n)})_{n \in \mathbb{N}}$ consisting of graphs where edges have been replaced by the two edge path. Here $a(0) = 0$ and $a(n) = a(n-1) + K_n$, where $(K_n)_{n \in \mathbb{N}}$ is a sequence of i.i.d. Geometric(p) random variables. Now $F_{a(n)}$ is obtained by taking $F_{a(n-1)}$ and replacing each edge by a model graph Γ_{K_n} where Γ_m consists of 2^{m-1} two-edge paths with endpoints identified.

Lemma 9.6. *The eigenvalues of $F_{a(n)}$ are $1 - \cos \frac{\pi k}{2^n}$ for $k = 0, 1, \dots, 2^n$.*

Proof. Now, for all m , the Laplacian of Γ_m is spectrally similar to that of the single edge F_0 , with the same function $R(z) = 4z - 2z^2$ and exceptional set $\{1\}$. Also the model graphs Γ_m satisfy the conditions of Lemma 4.7 of [36] so we can use that lemma to see that $F_{a(n)}$ is spectrally similar to $F_{a(n-1)}$ with this function and exceptional set.

This also applies to the sequence of paths of length 2^n (i.e. with 2^n edges) - indeed this is the special case where $p = 1$. So we see that the eigenvalues of $F_{a(n)}$ are the same as those of the path of length 2^n , i.e. they are $1 - \cos \frac{\pi k}{2^n}$ for $k = 0, 1, \dots, 2^n$ (see [11]). \square

We will now investigate the multiplicities of these eigenvalues.

9.2.1 The multiplicities of the eigenvalues

We consider the multiplicities of the various eigenvalues of the Laplacian of $F_{a(n)}$.

In what follows, e_n is the number of edges of F_n and v_n the number of vertices.

If we consider an m th generation eigenvalue to be one which is a Dirichlet-Neumann eigenvalue of $F_{a(m)}$ but not of $F_{a(m-1)}$, then, for $n \geq m$, the spectral similarity results tell us that the multiplicity of a given m th generation eigenvalue in the spectrum of $F_{a(n)}$ is the same as the multiplicity of the eigenvalue

1 (the only first generation eigenvalue) in $F_{a(n-m+1)}$, so in what follows we consider the multiplicity of the eigenvalue 1.

We know that $F_{a(n)}$ is spectrally similar to $F_{a(n-1)}$ with function $R(z) = 4z - 2z^2$ and exceptional set $\{1\}$. As $z = 1$ if and only if $R(z) = 2$, and $R(z) = \zeta$ has two distinct real roots for all $\zeta \in [0, 2)$, the multiplicity of eigenvalues which are not 1 is $2(v_{a(n-1)} - 1)$. So we will consider the ratio

$$\frac{2(v_{a(n-1)} - 1)}{v_{a(n)}}.$$

Note that

$$\begin{aligned} e_{a(n)} &= 2^{K_n} e_{a(n-1)} \\ v_{a(n)} &= v_{a(n-1)} + 2^{K_n-1} e_{a(n-1)} \end{aligned}$$

and hence

$$\begin{aligned} e_{a(n)} &= 2^{\sum_{i=1}^n K_i} \\ v_{a(n)} &= 2 + \sum_{i=1}^n 2^{\sum_{j=1}^i K_j}. \end{aligned}$$

So

$$\begin{aligned} \frac{v_{a(n-1)}}{v_{a(n)}} &= \frac{2 + \sum_{i=1}^{n-1} 2^{\sum_{j=1}^i K_j}}{2 + \sum_{i=1}^n 2^{\sum_{j=1}^i K_j}} \\ &= \frac{2 + \sum_{i=1}^{n-1} 2^{\sum_{j=1}^i K_j}}{2 + \sum_{i=1}^{n-1} 2^{\sum_{j=1}^i K_j} + 2^{\sum_{j=1}^n K_j}} \\ &= \left(1 + \frac{2^{\sum_{j=1}^n K_j}}{2 + \sum_{i=1}^{n-1} 2^{\sum_{j=1}^i K_j}} \right)^{-1} \\ &= \left(1 + \frac{1}{2^{-\sum_{j=1}^n K_j} + \sum_{i=1}^{n-1} 2^{\sum_{j=1}^i K_{n+1-i}}} \right)^{-1} \end{aligned}$$

So the limiting distribution of the ratio $\frac{2(v_{a(n-1)}-1)}{v_{a(n)}}$ is that of

$$\left(1 + \frac{1}{\sum_{i=1}^{\infty} 2^{-\sum_{j=1}^i K_j}} \right)^{-1}$$

so the limiting proportion of eigenvalues which are 1 is

$$1 - 2 \left(1 + \frac{1}{\sum_{i=1}^{\infty} 2^{-\sum_{j=1}^i K_j}} \right)^{-1}$$

$$= \frac{1 - \kappa_1}{1 + \kappa_1}$$

where

$$\kappa_1 = \sum_{i=1}^{\infty} 2^{-\sum_{j=1}^i K_j}$$

which is a random number in $[0, 1]$ obtained by considering an infinite sequence of i.i.d. Bernoulli random variables as a binary expansion.

A similar result holds when the random variables K_n are not geometric, but the digits of the random binary number κ_1 will no longer be independent.

9.2.2 More general results

When we consider more general homogeneous random fractal graphs, as defined in Definition 1.3, we will not generally have the property that the exceptional set and the function $R(z)$ will be the same for each model graph Γ_k . For each model graph Γ_k , we assume Γ_k to be spectrally similar to the cell F_0 (which is isomorphic to the complete graph on M vertices.) with function $R_k(z)$ and exceptional set \mathcal{E}_k .

Lemma 4.7 of [36] tells us that it is sufficient for each model graph Γ_k to be symmetric with respect to its boundary i.e. that each bijection $\partial\Gamma_k \rightarrow \partial\Gamma_k$ can be extended to a graph automorphism $\Gamma_k \rightarrow \Gamma_k$.

As in Definition 1.3, we let Γ_{Z_n} be the model graph replacing cells in the formation of F_n . We also let \mathcal{E} be the set of eigenvalues of the cell F_0 .

Then the spectral similarity tells us that the spectrum of F_n is contained in

the set

$$\mathcal{E}_{\Gamma_{z_n}} \cup R_{\Gamma_{z_n}}^{-1}(\mathcal{E}_{\Gamma_{z_{n-1}}}) \cup R_{\Gamma_{z_n}}^{-1} R_{\Gamma_{z_{n-1}}}^{-1}(\mathcal{E}_{\Gamma_{z_{n-2}}}) \cup \cdots \cup R_{\Gamma_{z_n}}^{-1} R_{\Gamma_{z_{n-1}}}^{-1} \cdots R_{\Gamma_{z_2}}^{-1}(\mathcal{E}_{\Gamma_{z_1}}) \cup R_{\Gamma_{z_n}}^{-1} R_{\Gamma_{z_{n-1}}}^{-1} \cdots R_{\Gamma_{z_1}}^{-1}(\mathcal{E}).$$

In the limit the distribution of this will be the same as that of the random set

$$\mathcal{E}_{\Gamma_{z_1}} \cup R_{\Gamma_{z_1}}^{-1}(\mathcal{E}_{\Gamma_{z_2}}) \cup R_{\Gamma_{z_1}}^{-1} R_{\Gamma_{z_2}}^{-1}(\mathcal{E}_{\Gamma_{z_3}}) \cup R_{\Gamma_{z_1}}^{-1} R_{\Gamma_{z_2}}^{-1} R_{\Gamma_{z_3}}^{-1}(\mathcal{E}_4) \cup \cdots .$$

An example is the random sequence of trees obtained by taking Γ_1 to be a two-edge path (where the boundary consists of the endpoints) and Γ_2 to be this path with an extra vertex, connected only to the midpoint (which is not part of the boundary). These graphs are shown in Figure 11.1. The functions $R_1(z)$ and $R_2(z)$ are $4z - 2z^2$ and $6z - 3z^2$ respectively, and the exceptional sets $\mathcal{E}_1 = 1$ and $\mathcal{E}_2 = \{1, 1 + 1/\sqrt{3}, 1 - 1/\sqrt{3}\}$.

Chapter 10

Eigenvalues of general non-homogeneous graphs

10.1 Relationships between homogeneous and non-homogeneous spectra

Throughout this section we consider graphs defined by Definition 1.3 under Assumption 1.1.

Consider a set of model graphs $\{\Gamma_r : r \in R\}$ and homogeneous and non-homogeneous random fractal graphs constructed according to some probability distribution on R , using the replacement construction.

Lemma 10.1. *If, for some $\lambda \in (0, 2)$, there is a positive probability that the homogeneous graph F_m has a Dirichlet-Neumann eigenfunction with eigenvalue λ , then there is also a positive probability that the non-homogeneous graph $G^{(n)}$ will have an eigenfunction with eigenvalue λ , for all $n \geq m$. Furthermore this probability converges to 1 as $n \rightarrow \infty$.*

Proof. For all $n \geq m$, $G^{(n)}$ consists of $c^{(n-m)}$ independent copies of $G^{(m)}$, connected together according to the structure of $G^{(n-m)}$. Now, if there is

a positive probability that F_m has a Dirichlet-Neumann eigenvalue λ , then there is also a positive probability that $G^{(m)}$ has a Dirichlet-Neumann eigenfunction with eigenvalue λ . Finally, as $c^{(n-m)} \rightarrow \infty$, we have the convergence to 1. \square

Note that non-homogeneous random fractal graphs can also have Dirichlet-Neumann eigenvalues which do not occur as Dirichlet-Neumann eigenfunctions of the corresponding homogeneous graph. The eigenvalues $1 \pm \sqrt{\frac{1}{6}}$ occurring in the graphs in Figure 8.2 are examples of this. In general this can occur when a subgraph appears which resembles a homogeneous random fractal graph with different model graphs.

The appearance of Dirichlet-Neumann eigenfunctions in non-homogeneous graphs appears to be due to “areas of homogeneity”, where neighbouring parts of the graph develop in the same way, which happens with positive probability due to the random fluctuations in the non-homogeneous construction.

10.2 Asymptotics for spectra of non-homogeneous fractal graphs

We now consider some asymptotics for the spectra of non-homogeneous random fractal graphs, as defined in Definition 1.3. Using the duplication construction, let $c_j^{(n,m)}$ be the number of m -cells in $G_j^{(n)}$ and let $c_j^{(n)}$ be the number of cells in $G_j^{(n)}$.

Lemma 10.2. *For fixed m , $\frac{c_1^{(n)}}{c_1^{(n,m)}}$ converges to a non-zero constant almost surely, as $n \rightarrow \infty$.*

Proof. The graph $G_1^{(n)}$ consists of $c_1^{(n,m)}$ m -cells, which are independent graphs with the same distribution as $G_1^{(m)}$. So the total number of cells is the sum of $c_1^{(n,m)}$ bounded (because the maximum number of vertices in an m -cell

is bounded) and non-zero independent identically distributed random variables, and as $c_1^{(n,m)} \rightarrow \infty$ as $n \rightarrow \infty$, the result follows from the Strong Law of Large Numbers. \square

We now consider an individual eigenvalue $\lambda \in [0, 2]$ and define the multiplicity of Dirichlet-Neumann eigenfunctions in $G_j^{(n)}$ with eigenvalue λ to be $l_{j,\lambda}^{(n)}$.

Further we define $l_{j,\lambda}^{(n,m)}$ to be the multiplicity of Dirichlet-Neumann eigenfunctions with eigenvalue λ in $G_j^{(n)}$ which are non-zero only in the interior of some m -cell. We will call these eigenfunctions *m-localised*.

It is obvious that, if $m > m'$, then $l_{j,\lambda}^{(n,m)} \geq l_{j,\lambda}^{(n,m')}$.

Lemma 10.3. *For each m , the multiplicity of m -localised eigenfunctions with eigenvalue λ per cell*

$$\frac{l_{1,\lambda}^{(n,m)}}{c_1^{(n)}}$$

converges to a constant $\tilde{l}_{\lambda,m}$, almost surely, as $n \rightarrow \infty$.

Proof. The numbers of m -localised eigenvalues localised on each m -cell are bounded independent and identically distributed random variables, so because the number of m -cells $c_1^{(n,m)} \rightarrow \infty$ as $n \rightarrow \infty$ we use the Strong Law of Large Numbers to obtain that

$$\frac{l_{1,\lambda}^{(n,m)}}{c_1^{(n)}}$$

converges to a constant almost surely.

We now use Lemma 10.2 to obtain the result. \square

The limit $\lim_{m \rightarrow \infty} \tilde{l}_{m,\lambda}$ provides an almost sure lower bound for the lim inf multiplicity per cell, $\liminf_{n \rightarrow \infty} \frac{l_{1,\lambda}^{(n)}}{c_n^{(1)}}$.

Lemma 10.4. *The ratio of the number of vertices to the number of cells, $\frac{v_1^{(n)}}{c_1^{(n)}}$, converges to a positive constant, \hat{v} , in probability.*

Proof. We switch to considering the replacement construction. The sequence of the numbers of cells in $G^{(n)}$, $(c^{(n)})_{n \in \mathbb{N}}$, forms a branching process with offspring distribution concentrated on $\mathbb{N} \setminus \{0\}$, and hence, for μ the mean of the offspring distribution,

$$\mu^{-n} c^{(n)} \rightarrow W,$$

almost surely, with W a random variable, by general branching process theory [4].

Now, the structure of the graphs implies that the number of vertices $v^{(n)}$ of $G^{(n)}$ satisfies

$$v^{(n)} = v^{(n-1)} + \sum_{j=1}^{c^{(n-1)}} v_{Z_j^{(n)}}$$

and so is the sum of Y_{n-1} independent and identically distributed random variables, where $Y_n = \sum_{m=1}^n c^{(m)}$, the total population of the branching process up to time n .

Now the total population Y_n satisfies $\mu^{-n} Y_n \rightarrow \frac{\mu}{\mu-1} W$, almost surely, as $n \rightarrow \infty$, so another application of the Strong Law of Large Numbers gives that

$$\frac{v^{(n)}}{c^{(n)}}$$

converges to a constant, almost surely, which, by converting back to the duplication construction, implies the result.

Because $\mathbb{E}v_{Z_j^{(n)}} > 0$, the constant $\hat{v} > 0$.

□

This gives us the following:

Corollary 10.5. *Given a duplication sequence $(G_1^{(n)})_{n \in \mathbb{N}}$, as defined in Definition 1.3, and some λ which occurs as a Dirichlet-Neumann eigenvalue of*

$G^{(m)}$, for some $m \in \mathbb{N}$, with strictly positive probability, we have a strictly positive deterministic lower bound $\hat{l}_{\infty, \lambda} > 0$ such that the limiting proportion

$$\liminf_{n \rightarrow \infty} \frac{l_{1, \lambda}^{(n)}}{v_1^{(n)}} \geq \hat{l}_{\infty, \lambda},$$

almost surely.

Proof. We set $\hat{l}_{\infty, \lambda} = \sup_m \tilde{l}_{\lambda, m} \hat{v}$. Because λ occurs as a Dirichlet-Neumann eigenvalue of $G^{(m)}$ with positive probability, $\tilde{l}_{\lambda, m} > 0$, and, by Lemma 10.4, $\hat{v} > 0$, we have $\hat{l}_{\infty, \lambda} > 0$. \square

In the special case where the number of cells in a 1-cell (model graph) is constant, we can go further.

Theorem 10.6. *Given a duplication construction of a non-homogeneous random fractal graph, where the model graph size $c_i = c$ for each i , the limiting proportion of eigenfunctions which are Dirichlet-Neumann with eigenvalue λ ,*

$$\frac{l_{1, \lambda}^{(n)}}{v_1^{(n)}},$$

converges to a constant, almost surely.

Proof. We note that the condition on model graph size implies that $c_j^{(n)} = c^n$ for all n, j . Now

$$l_{1, \lambda}^{(n)} \geq \sum_{j=1} c l_{j, \lambda}^{(n-1)}$$

and

$$v_1^{(n)} \leq \sum_{j=1} c v_j^{(n-1)}.$$

Now, we define σ -algebras $\mathcal{F}_n = \sigma(G_1^{(m)}; m \geq n)$. By the symmetry of the input graphs $G_j^{(n-1)}$ given \mathcal{F}_n , $(l_{1, \lambda}^{(n)}/c^n)_{n \in \mathbb{N}}$ is a backwards supermartingale with respect to (\mathcal{F}_n) , and similarly $(v_1^{(n)}/c^n)_{n \in \mathbb{N}}$ is a positive backwards submartingale.

Also, $l_{1,\lambda}^{(n)} \leq v_1^{(n)}$, and hence backwards martingale theory implies that

$$\frac{l_{1,\lambda}^{(n)}}{c^n} \rightarrow \tilde{l}_\lambda$$

and

$$\frac{v_1^{(n)}}{c^n} \rightarrow \tilde{v},$$

almost surely, where \tilde{l}_λ and \tilde{v} are random variables.

So, for $\epsilon > 0$,

$$\left| \frac{l_{1,\lambda}^{(n)}}{c^n} - \tilde{l}_\lambda \right| < \epsilon$$

for $n > n_\epsilon$ (where n_ϵ is random), almost surely. Hence, for all $n > n_\epsilon$,

$$\left| \frac{l_{1,\lambda}^{(n)}}{c^n} - \frac{l_{1,\lambda}^{(n-1)}}{c^{n-1}} \right| < 2\epsilon,$$

almost surely.

But

$$\left| \frac{l_{1,\lambda}^{(n)}}{c^n} - \frac{l_{1,\lambda}^{(n-1)}}{c^{n-1}} \right| \geq \sum_{j=2}^n \frac{l_{j,\lambda}^{(n-1)}}{c^{n-1}} - l_{1,\lambda}^{(n-1)} \left(\frac{c-1}{c^n} \right).$$

Hence $\mathbb{P}(l_2^{(n-1)} > \tilde{l}_\lambda + \epsilon) \rightarrow 0$ as $n \rightarrow \infty$, for all $\epsilon > 0$. As $l_2^{(n-1)}$ and $l_1^{(n-1)}$ are independent and identically distributed, this shows that the limit must be constant.

□

The series-parallel graph satisfies the conditions of Theorem 10.6, but many other examples do not, for example the randomised gasket of Figure 1.3.

Chapter 11

Some further examples

We consider some further examples of spectra of random fractal graphs.

11.1 A random tree

We consider random fractal graph structures with the two model graphs shown in Figure 11.1. The graphs resulting from this are random trees.

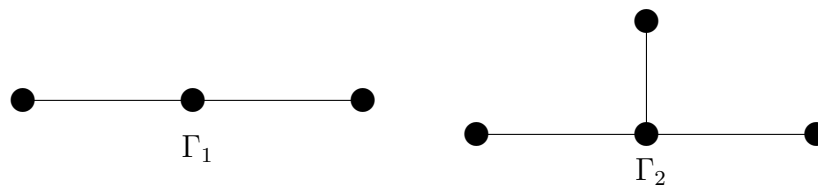


Figure 11.1: The model graphs for the random fractal tree

For the homogeneous version, spectral similarity applies with the functions $R_1(z) = 4z - 2z^2$ and $R_2(z) = 6z - 3z^2$, exceptional sets $\mathcal{E}_1 = \{1\}$, $\mathcal{E}_2 = \{1, 1 + \frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}}\}$. So unlike the series-parallel case the functions are different

for the two model graphs.

Considering the non-homogeneous case, we put a probability p on Γ_1 and $1 - p$ on Γ_2 . The case $p = 0$ again gives paths of length 2^n , while the case $p = 1$ gives a tree whose eigenvalues can be studied by using the spectral similarity methods of [36]. Because the degree of the map $R_2(z)$ is 2 and the graph $G^{(n)}$ has 3^n vertices, not all eigenvalues of $G^{(n)}$ are obtained by spectral decimation from those of $G^{(n)}$. The remainder must (by Theorem 3.6 of [36]) be contained within the exceptional set. So we expect high multiplicity on some values within the exceptional set.

The histograms in Figures 11.2, 11.3 and 11.4 show samples of the eigenvalues for the non-homogeneous case, for small n . For $p = 0.9$ we seem to get a largely point spectrum with large concentrations on the Dirichlet-Neumann eigenvalues of the deterministic $p = 1$ case, while for $p = 0.5$ and $p = 0.1$ there seem to be fewer Dirichlet-Neumann eigenvalues, in a similar way to the series-parallel graph for large p .

This is expected, because at $p = 0.9$ there will be a relatively high probability of a cell being identical to a corresponding cell in the $p = 1$ case, i.e. a deterministic fractal graph. This is less noticeable as we approach $p = 0$ because this deterministic limit (the path of length 2^n) does not have Dirichlet-Neumann eigenfunctions.

The symmetry of these spectra is due to the fact that the random trees are bipartite, which implies that if λ is an eigenvalue then so is $2 - \lambda$, with the same multiplicity [11].

11.2 Non-symmetric model graphs

The conditions on the model graphs in Section 9.2.2, and in [36], are rather strong. Many examples of self-similar fractals are related to graphs which do not satisfy the symmetry condition. These include the Lindström snowflake (Example 6.4 of [36]), and the pentagasket, studied in [1], where families of eigenvalues and eigenfunctions are described, and the eigenvalues are shown

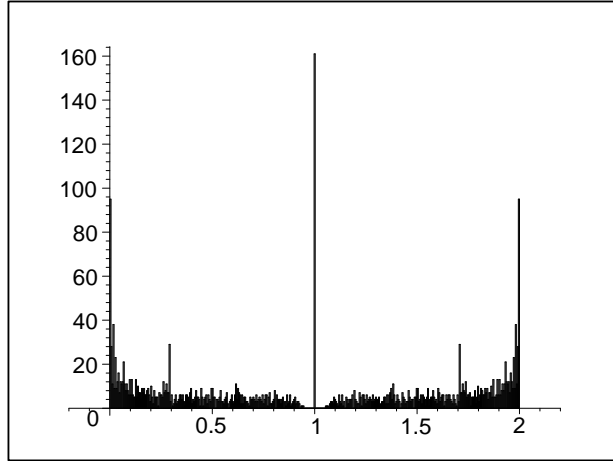


Figure 11.2: Histogram of eigenvalues of random tree $G^{(6)}$ with $p = 0.1$, 30 graphs sampled

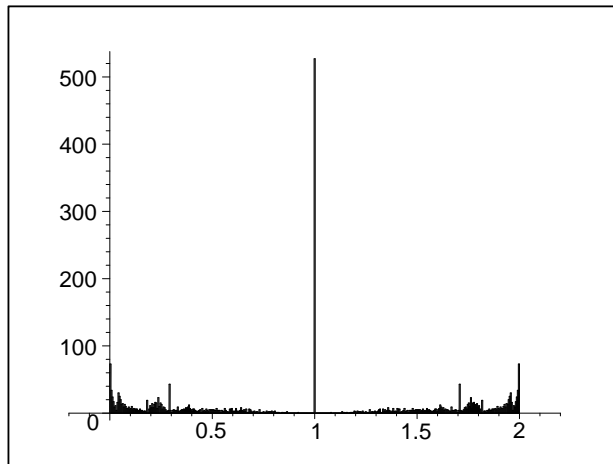


Figure 11.3: Histogram of eigenvalues of random tree $G^{(5)}$ with $p = 0.5$, 30 graphs sampled

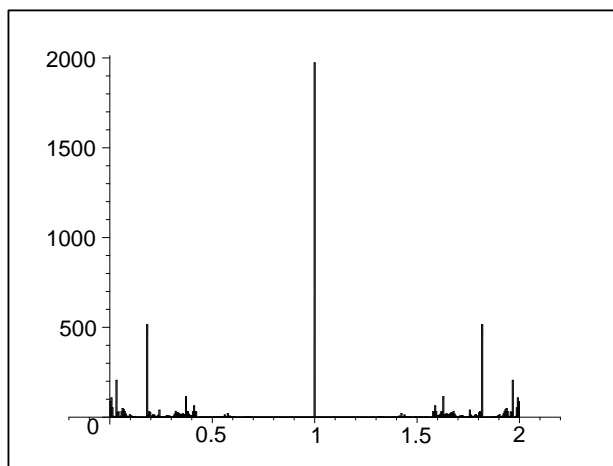


Figure 11.4: Histogram of eigenvalues of random tree $G^{(5)}$ with $p = 0.9$, 30 graphs sampled

to have high multiplicity, even though the spectral decimation method does not work.

A simple example of a graph that does not satisfy the conditions is one of the model graphs in the network of Section 3.3.4. We will consider two random self-similar graphs which use this as a model graph.

To precisely specify the construction of the graphs (see Definition 1.3), we need to make it clear which orientation the model graph has. There are several ways of doing this, but the following (replacement) construction has interesting properties:

- $G^{(0)}$ consists of two vertices $\{1, 2\}$ and a single edge between them.
- Label each vertex added at stage n with an integer higher than any used to label a vertex of $G^{(n-1)}$.
- Consider each edge in $E(G^{(n)})$ as $a \leftrightarrow b$ with $a < b$ and replace it with a model graph, by identifying boundary vertices 1 and 2 of the model graph with vertices a and b respectively.

- The model graphs may be random or deterministic.

We will consider the model graphs Γ_1 and Γ_2 shown in Figure 11.5, so that Γ_2 is the mirror image of Γ_1 .

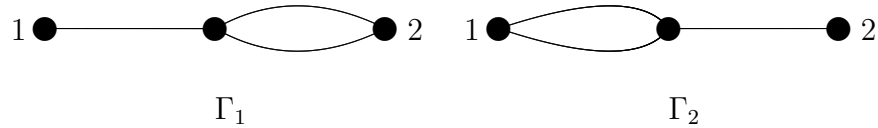


Figure 11.5: The asymmetric model graphs

These graphs are related to those considered in [15], where the resistance (which does not depend on the orientation) was studied.

We first consider the deterministic sequence where the model graph is always Γ_1 . The graph $G^{(n)}$ here has $v^{(n)}$ vertices and $e^{(n)}$ edges where $v^{(0)} = 2$, $e^{(0)} = 1$ and $e^{(n)} = 3e^{(n-1)}$, $v^{(n)} = v^{(n-1)} + e^{(n-1)}$. Hence $e^{(n)} = 3^n$ and $v^{(n)} = \frac{1}{2}(3^n + 3)$.

We define maps $f_i : V(G^{(n-1)}) \rightarrow V(G^{(n)})$, $i = 1, 2, 3$ mapping each vertex of $G^{(n-1)}$ to the corresponding vertex in each $(n - 1)$ -cell. We will label these so that f_1 and f_2 correspond to the two parallel cells.

Proposition 11.1. *The graph $G^{(n)}$ has at least $\frac{1}{2}(3^n + 3) - 2^n - 1$ linearly independent Dirichlet-Neumann eigenvalues.*

Proof. The model graph contains parallel edges, so $G^{(n)}$ contains a subgraph consisting of two copies of $G^{(n-1)}$ with their boundary points identified as in Lemma 8.2. This gives $v^{(n-1)} - 2$ eigenfunctions. For each eigenfunction x obtained thus, we have $x(f_1(v)) = -x(f_2(v))$ and $x(f_3(v)) = 0$ for each $v \in V(G^{(n-1)})$.

Furthermore, given a Dirichlet-Neumann eigenfunction x of $G^{(n-1)}$ we can obtain 3 Dirichlet-Neumann eigenfunctions x_1, x_2, x_3 of $G^{(n)}$ by extending

them from $(n - 1)$ -cells to the whole graph i.e.

$$x_i(f_j(v)) = \delta_{ij}x(v) \text{ for all } v \in V(G^{(n-1)}), i, j = 1, 2, 3.$$

However, we only obtain 2 linearly independent eigenfunctions, because the linear combination $x_1 - x_2$ is of the form obtained using Lemma 8.2.

Hence, if $l^{(n)}$ is the number of Dirichlet-Neumann eigenvalues constructed by these methods, it satisfies

$$l^{(n)} = 2l^{(n-1)} + \frac{1}{2}(3^{n-1} + 3) - 2$$

and $l^{(2)} = 1$, which gives the result. \square

Let the total number of Dirichlet-Neumann eigenfunctions of $G^{(n)}$ be $l^{(n)} + \hat{l}^{(n)}$, so that $\hat{l}^{(n)}$ is the number which are not constructed by the methods of Proposition 11.1

For this graph, we can also describe a set of non-Dirichlet-Neumann eigenfunctions.

Proposition 11.2. *The graph $G^{(n)}$ has $2^n + 1$ linearly independent eigenfunctions which are not Dirichlet-Neumann.*

Proof. We consider the set of eigenfunctions x satisfying $x(3) = 0$ and, for each $v \in V(G^{(n-1)})$, $x(f_1(v)) = x(f_2(v)) = \frac{1}{2}x(f_3(v))$, a $(v^{(n-1)} - 1)$ -dimensional subspace. This is preserved by the Laplacian of $G^{(n)}$, so we can find $v^{(n-1)} - 1 = \frac{1}{2}(3^{n-1} + 1)$ linearly independent eigenfunctions satisfying these properties.

To exclude those which are Dirichlet-Neumann, this is equivalent to the condition that $x(1) = x(2) = 0$. So a Dirichlet-Neumann eigenfunction satisfying the above conditions reduces to a Dirichlet-Neumann eigenfunction on each $(n - 1)$ -cell. Hence there are $\frac{1}{2}(3^{n-1} + 1) - l^{(n-1)} - \hat{l}^{(n-1)} = 2^{n-1} - \hat{l}^{(n-1)}$ non-Dirichlet-Neumann eigenfunctions satisfying the above conditions.

Now, given any eigenfunction x of $G^{(n-1)}$, we can extend it to an eigenfunction x' of $G^{(n)}$ by setting $x'(1) = \sqrt{2}x(1)$, $x'(2) = x(1)$, $x'(3) = \sqrt{3}x(2)$,

$x'(f_1(v)) = x'(f_2(v)) = x'(f_3(v)) = x(v)$ for $v \geq 3$, from the structure of the graph. This means that each of the eigenfunctions constructed in Proposition 11.2 for $G^{(n-1)}$ can be extended to a non-Dirichlet-Neumann eigenfunction of $G^{(n)}$, which will be linearly independent of those already found (because $x'(3) \neq 0$). Inductively, this also applies to those constructed for $G^{(n-m)}$, $m > 1$.

The graphs are bipartite, so eigenfunctions (which are zero nowhere) for eigenvalues 0 and 2 exist as described in [11].

Hence the total number of linearly independent non-Dirichlet-Neumann eigenfunctions is at least $2^n + 1 - \sum_{m=0}^{n-1} \hat{l}^{(m)}$. But we know that there are exactly $\frac{1}{2}(3^n + 3)$ linearly independent eigenfunctions. Hence

$$\frac{1}{2}(3^n + 3) \geq 2^n + 1 - \sum_{m=0}^{n-1} \hat{l}^{(m)} + \frac{1}{2}(3^n + 3) - 2^n - 1 + \hat{l}^{(n)}$$

and so $\hat{l}^{(n)} \leq \sum_{m=0}^{n-1} \hat{l}^{(m)}$. But $\hat{l}^{(m)} = 0$ for $m \leq 2$, and hence for all m .

Hence the eigenfunctions constructed are all that exist, and there are $2^n + 1$ linearly independent non-Dirichlet-Neumann ones, the remainder being Dirichlet-Neumann. \square

We note that these proofs also apply to the (different) fractal graph constructed using the model graph Γ_2 .

We now consider two random non-homogeneous fractal graphs related to this graph. First, we consider the sequence obtained with the model graphs Γ_1 and Γ_2 above, with probabilities p and $1 - p$ respectively. This can be considered as using the same model graph but with a random orientation for each edge replacement. Samples of eigenvalues for $p = 0.2$ and $p = 0.5$ are shown in Figures 11.6 and 11.7. In this example, the first level of localised eigenvalues (which are all 1) is not affected by the random orientation, and hence the multiplicity of 1 is large for both graphs. Apart from this, however, there is more evidence of eigenvalues having high multiplicity for the $p = 0.2$ graphs. As for the random trees, the graphs are bipartite, so the spectra are symmetric.

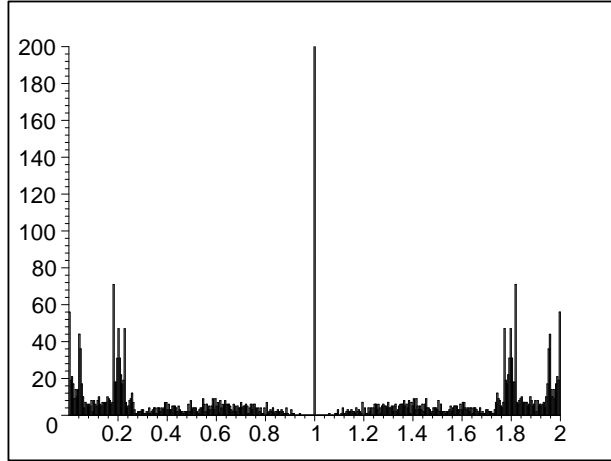


Figure 11.6: Histogram of eigenvalues of random orientation graph $G^{(5)}$ with $p = 0.2$, 30 graphs sampled. The total multiplicity of 1 is 1170.

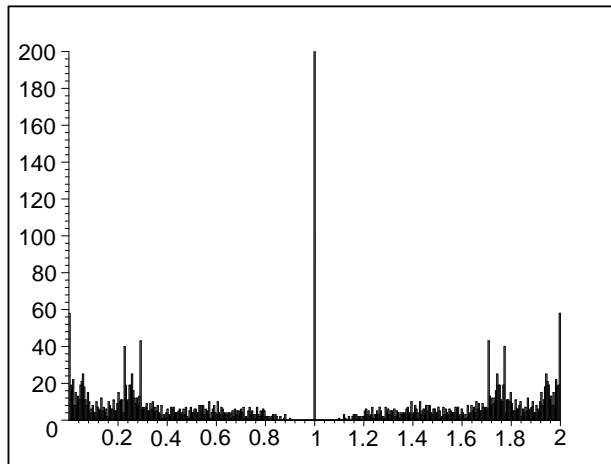


Figure 11.7: Histogram of eigenvalues of random orientation graph $G^{(5)}$ with $p = 0.5$, 30 graphs sampled. The total multiplicity of 1 is 1170.

Finally we consider the graphs discussed in Section 3.3.4. The $p = 1$ extreme is discussed above, while spectral similarity applies to the $p = 0$ extreme (which is not bipartite) with the simple function $R(z) = 2z$. The histograms in Figures 11.8, 11.9 and 11.10 show samples for $p = 0.2, 0.5$ and 0.8 respectively. As before, the examples with high probability on one model graph resemble the spectra of the deterministic graphs associated with that model graph.

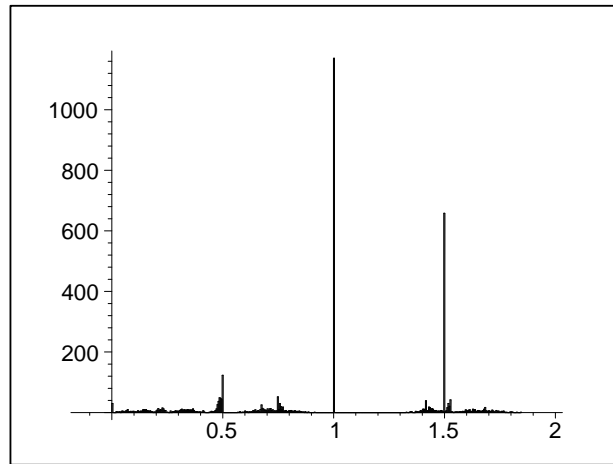


Figure 11.8: Histogram of eigenvalues of graph 3.3.4 $G^{(5)}$ with $p = 0.2$, 30 graphs sampled

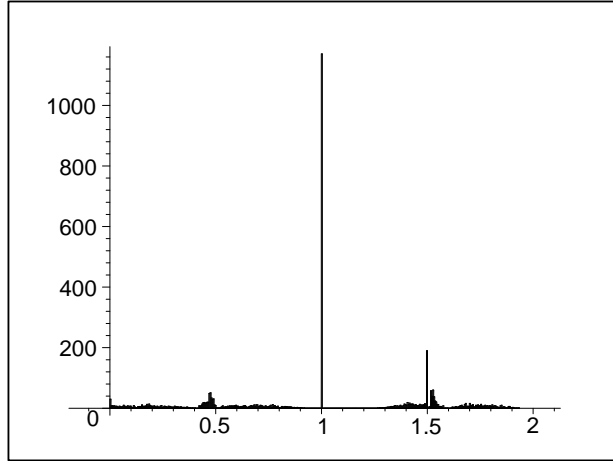


Figure 11.9: Histogram of eigenvalues of graph 3.3.4 $G^{(5)}$ with $p = 0.5$, 30 graphs sampled

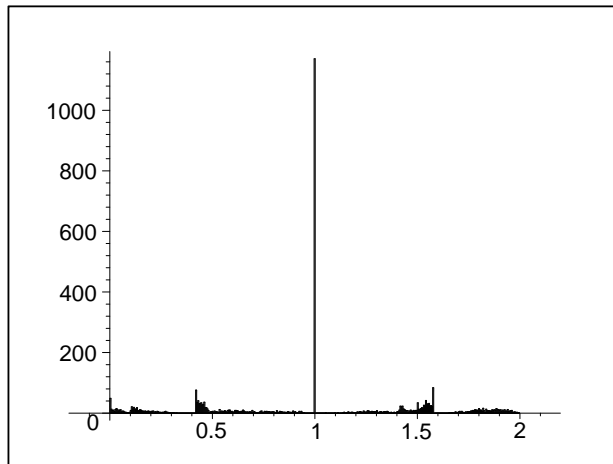


Figure 11.10: Histogram of eigenvalues of graph 3.3.4 $G^{(5)}$ with $p = 0.8$, 30 graphs sampled

Bibliography

- [1] B. Adams, S. A. Smith, R. S. Strichartz, and A. Teplyaev. The spectrum of the Laplacian on the pentagasket. In *Trends in Mathematics: Fractals in Graz 2001*, pages 1–24. Birkhauser, 2003.
- [2] N. Alon and M. Naor. Coin-flipping games immune against linear sized coalitions. *SIAM J. Comput.*, 22:403–417, 1993.
- [3] N. Alon and M. O. Rabin. Biased coins and randomized algorithms. In S. Micali, editor, *Advances in Computing Research 5: Randomness and Computation*, pages 499–507. JAI Press, Greenwich, Connecticut, 1989.
- [4] K. B. Athreya and P. E. Ney. *Branching processes*. Springer-Verlag, Berlin, 1972.
- [5] M. T. Barlow and B. M. Hambly. Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets. *Ann. Inst. Henri Poincaré*, 33:531–557, 1997.
- [6] L. Bartholdi and R. I. Grigorchuk. Spectra of non-commutative dynamical systems and graphs related to fractal groups. *C. R. Acad. Sci. Paris*, 331:429–434, 2000.
- [7] J. Bernasconi. Real renormalization of bond disordered conductance lattices. *Phys. Rev. B*, 18:2185–2191, 1978.
- [8] B. Bollobás. *Random Graphs*. Academic Press, London, 1985.
- [9] R. Boppana and B. Narayanan. The biased coin problem. In *Proceedings of the 35th ACM Symposium on Theory of Computing*, pages 252–257. AGM Press, New York, 1993.

- [10] R. Boppana and B. Narayanan. Perfect-information leader election with optimal resilience. *SIAM J. Comput.*, 29:1304–1320, 2000.
- [11] F. R. K. Chung. *Spectral Graph Theory*. Number 92 in CBMS Regional Conference Series. AMS, Providence, Rhode Island, 1997.
- [12] D. M. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs, Theory and Applications*. Academic Press, London, 1980.
- [13] R. Derrida. Pure and random models of statistical mechanics on hierarchical lattices. In K. Osterwalder and R. Stora, editors, *Critical Phenomena, Random Systems and Gauge Theories*, pages 989–999. North-Holland, Amsterdam, 1986.
- [14] R. Durrett and T. M. Liggett. Fixed points of the smoothing transformation. *Zeitschrift für Wahrscheinlichkeitstheorie*, 64:275–301, 1983.
- [15] C. D. Essoh and J. Bellisard. Resistance and fluctuation of a fractal network of random resistors: a non-linear law of large numbers. *J. Phys. A*, 22:4537–4548, 1989.
- [16] K. J. Falconer. Random fractals. *Math. Proc. Camb. Phil. Soc.*, 100:559–582, 1986.
- [17] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley and Sons, Chichester, 1990.
- [18] I. J. Farkas, I. Derényi, A-L. Barabási, and T. Vicsek. Spectra of “real-world” graphs: Beyond the semi circle law. *Phys. Rev. E*, 64:026704, 2001.
- [19] S. Graf. Statistically self-similar fractals. *Prob. Theor. Related Fields*, 74:357–392, 1987.
- [20] R. Griffiths and M. Kaufman. Spin systems on hierarchical lattices. *Phys. Rev. B*, 26:5022–5032, 1982.
- [21] B. M. Hambly. Brownian motion on a homogeneous random fractal. *Prob. Theory Rel. Fields*, 94:1–38, 1992.
- [22] B. M. Hambly. Brownian motion on a random recursive Sierpiński gasket. *Ann. Prob.*, 25:1059–1102, 1997.

- [23] B. M. Hambly. On the asymptotics of the eigenvalue counting function for random recursive Sierpiński gaskets. *Prob. Theory Rel. Fields*, 117:221–247, 2000.
- [24] B. M. Hambly and V. Metz. The homogenization problem for the Vicsek set. *Stoc. Proc. Appl.*, 76:167–190, 1998.
- [25] B. M. Hambly and N. O’Connell. A law of large numbers for random hierarchical sequences. Technical Report HPL-BRIMS-2000-13, BRIMS, 2000.
- [26] J. E. Hutchinson and L. Rüschemdorf. Random fractal measures via the contraction method. *Indiana Univ. Math. J.*, 47:471–487, 1998.
- [27] J. E. Hutchinson and L. Rüschemdorf. Random fractals and probability metrics. *Adv. Appl. Prob.*, 32:925–947, 2000.
- [28] J. Jona-Lasinio. The renormalization group: A probabilistic view. *Nuovo Cimento B*, 26:99–119, 1975.
- [29] J. H. Jordan. Almost sure convergence for iterated functions of independent random variables. *Ann. App. Prob.*, 12:985–1000, 2002.
- [30] J. Kigami. *Analysis on Fractals*. Cambridge University Press, 2001.
- [31] T. Kumagai and S. Kusuoka. Homogenization on nested fractals. *Probab. Theory Relat. Fields*, 104:375–398, 1996.
- [32] D. L. Li and T. D. Rogers. Asymptotic behavior for iterated functions of random variables. *Ann. App. Prob.*, 9:1175–1201, 1999.
- [33] T. Lindstrøm. Brownian motion on nested fractals. *Mem. Amer. Math. Soc.*, 420, 1990.
- [34] Q. Liu. Fixed points of a generalized smoothing transformation and applications to the branching random walk. *Adv. Appl. Prob.*, 30:85–112, 1998.
- [35] L. Malozemov and A. Teplyaev. Pure point spectrum of the Laplacians on fractal graphs. *J. Func. Anal.*, 129:390–405, 1994.

- [36] L. Malozemov and A. Teplyaev. Self-similarity, operators and dynamics. Preprint, 2001.
- [37] R. D. Mauldin and S. C. Williams. Random recursive constructions: asymptotic geometric and topological properties. *Trans. Am. Math. Soc.*, 295:325–346, 1986.
- [38] E. Moore and C. E. Shannon. Reliable circuits using less reliable relays, i. *J. Franklin Inst.*, 262:191–208, 1956.
- [39] E. Moore and C. E. Shannon. Reliable circuits using less reliable relays, ii. *J. Franklin Inst.*, 262:281–297, 1956.
- [40] W. I. Newman, A. M. Gabrielov, T. A. Durand, S. L. Phoenix, and D. L. Turcotte. An exact renormalization model for earthquakes and material failure. *Physica D*, 77:200–216, 1994.
- [41] R. Rammal and G. Toulouse. Random walks on fractal structures and percolation clusters. *J. Physique Letters*, 44:L13–L22, 1983.
- [42] M. C. Reed and B. Simon. *Methods of Modern Mathematical Physics - 1: Functional Analysis*. Academic Press, London, 1972.
- [43] C. Sabot. Pure point spectrum for the Laplacian on unbounded nested fractals. *J. Funct. Anal.*, 173:497–524, 2000.
- [44] A. Schenkel, J. Wehr, and P. Wittwer. Computer-assisted proofs for fixed point problems in Sobolev spaces. *Elec. J. Math. Phys.*, 6(3), 2000.
- [45] T. Schlösser and H. Spohn. Sample-to-sample fluctuations in the conductivity of a disordered medium. *J. Stat. Phys.*, 69:955–967, 1992.
- [46] T. Shima. On eigenvalue problems for Laplacians on p.c.f. self-similar sets. *Japan. J. Indust. Appl. Math.*, 13:1–23, 1996.
- [47] I. Ya. Shneiberg. Hierarchical sequences of random variables. *Theory Probab. Appl.*, 31:137–141, 1986.
- [48] R. B. Stinchcombe and B. P. Watson. Renormalization group approach for percolation conductivity. *J. Phys. C*, 9:3221–3247, 1976.

- [49] R. S. Strichartz. Fractals in the large. *Canadian J. of Math.*, 50:638–657, 1998.
- [50] A. Teplyaev. Spectral analysis on infinite Sierpiński gaskets. *J. Funct. Anal.*, 159:537–567, 1998.
- [51] J. Wehr. A strong law of large numbers for iterated functions of independent random variables. *J. Stat. Phys.*, 86:1373–1384, 1997.
- [52] J. Wehr and J-M. Woo. Random conductance on hierarchical lattice. Preprint, 2000.
- [53] J. H. Wilkinson. *The Algebraic Eigenvalue Problem*. Oxford University Press, 1965.